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In any Δ ABC with p_a, p_b, p_c
$\to$ Spieker cevians, the following relationship holds :
$$(3p_a + w_a).AI + (3p_b + w_b).BI + (3p_c + w_c).CI \ge 2(a^2 + b^2 + c^2)$$

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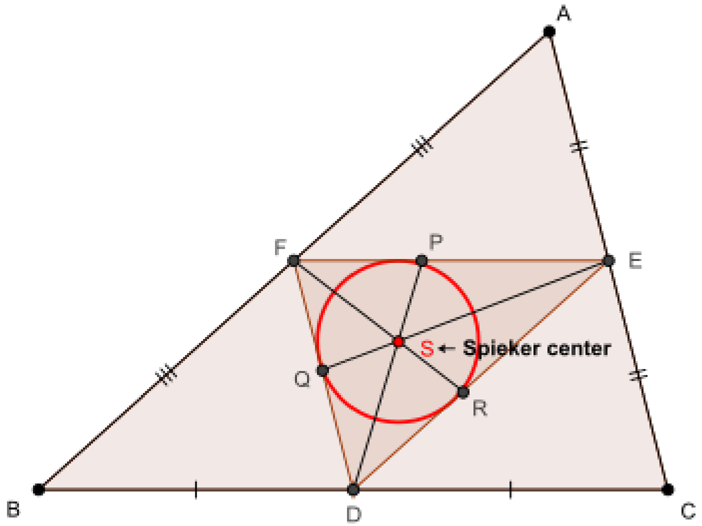

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES},b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Also, } p_a^2-m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)-m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2-m_a^2 \overset{(\blacksquare\blacksquare)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \Rightarrow p_a w_a \geq m_a w_a \overset{\text{Lascu + A-G}}{\geq}$$

$$s(s-a) \to (3) \text{ Also, } p_a^2-m_a^2 \overset{\text{via } (\blacksquare\blacksquare)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \Rightarrow$$

$$p_a^2 = s(s-a)+\frac{(b-c)^2}{4}+\frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} = s(s-a)+\frac{(b-c)^2(12s^2+4sa)}{4(2s+a)^2}$$

$$\Rightarrow p_a^2 \overset{(\blacksquare\blacksquare\blacksquare)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } (3p_a+w_a)^2 = 9p_a^2+w_a^2+6p_a w_a \overset{\text{via } (\blacksquare\blacksquare\blacksquare) \text{ and } (3)}{\geq}$$

$$9s(s-a)+\frac{9s(3s+a)(b-c)^2}{(2s+a)^2}+s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}+6s(s-a)$$

$$\overset{?}{\geq} 16m_a^2 = 16s(s-a)+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)(b-c)^2}{(2s-a)^2}+4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)+4(2s-a)^2}{(2s-a)^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 20t^4-36t^3+13t^2+5t-2 \overset{?}{\geq} 0 \quad \left(t=\frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\left((t-1)(20t^2+4t+1)+3\right) \overset{?}{\geq} 0 \to \text{true (strict)} \because t>1$$

$$\therefore 3p_a+w_a \geq 4m_a \textit{ and analogs} \Rightarrow \sum_{\text{cyc}}\left((3p_a+w_a).AI\right) \geq 4r\sum_{\text{cyc}}\left(\frac{m_a}{\sin\frac{A}{2}}\right)$$

$$\overset{\text{Lascu + A-G}}{\geq} 4r\sum_{\text{cyc}}\left(\frac{\frac{b+c}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}}\right) = 2rs\sum_{\text{cyc}}\left(\frac{b+c}{r_a}\right) = \frac{2rs}{rs^2}\sum_{\text{cyc}}\left((b+c)s(s-a)\right)$$

$$=2\sum_{cyc}\left((2s-a)(s-a)\right)=2\left(2s\sum_{cyc}(s-a)-\left(s(2s)-\sum_{cyc}a^2\right)\right)=2\sum_{cyc}a^2$$

$$\therefore (3p_a+w_a).AI+(3p_b+w_b).BI+(3p_c+w_c).CI\ge 2\left(a^2+b^2+c^2\right)$$

$\forall\,\Delta\,ABC$,'' ='' iff $\Delta\,ABC$ is equilateral (QED)