ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{r_b+r_c}{r_a+2m_a}+\frac{r_c+r_a}{r_b+2m_b}+\frac{r_a+r_b}{r_c+2m_c}\ge 2$$

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$$\frac{r_b+r_c}{r_a+2m_a}+\frac{r_c+r_a}{r_b+2m_b}+\frac{r_a+r_b}{r_c+2m_c}=\sum_{cyc}\frac{(r_b+r_c)^2}{r_a(r_b+r_c)+2m_a(r_b+r_c)}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}(r_b+r_c)\right)^2}{\sum_{cyc}r_a(r_b+r_c)+2\sum_{cyc}m_a(r_b+r_c)}=\frac{4(4R+r)^2}{2s^2+2\sum_{cyc}m_a(r_b+r_c)}\overset{?}{\ge}2$$

$$\Leftrightarrow (4R+r)^2-s^2\underset{(*)}{\overset{?}{\ge}}\sum_{cyc}m_a(r_b+r_c)$$

$$\text{Now, }\sum_{cyc}m_a(r_b+r_c)\overset{\text{CBS}}{\le}\sqrt{\sum_{cyc}m_a^2}.\sqrt{\sum_{cyc}(r_b+r_c)^2}$$

$$=\sqrt{\frac{3}{4}.2(s^2-4Rr-r^2)}.\sqrt{2\left(\left((4R+r)^2-2s^2\right)+s^2\right)}$$

$$=\sqrt{3(s^2-4Rr-r^2)((4R+r)^2-s^2)}\overset{?}{\le}(4R+r)^2-s^2$$

$$\Leftrightarrow 3\left(s^2-4Rr-r^2\right)\overset{?}{\le}(4R+r)^2-s^2\Leftrightarrow s^2\overset{?}{\le}4R^2+5Rr+r^2$$

$$\Leftrightarrow\left(s^2-\left(4R^2+4Rr+3r^2\right)\right)-r(R-2r)\overset{?}{\le}0\to\text{true}\because s^2-\left(4R^2+4Rr+3r^2\right)$$

$$\overset{\text{Gerretsen}}{\le}0\,and\,-r(R-2r)\overset{\text{Euler}}{\le}0\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{r_b+r_c}{r_a+2m_a}+\frac{r_c+r_a}{r_b+2m_b}+\frac{r_a+r_b}{r_c+2m_c}\ge 2$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)