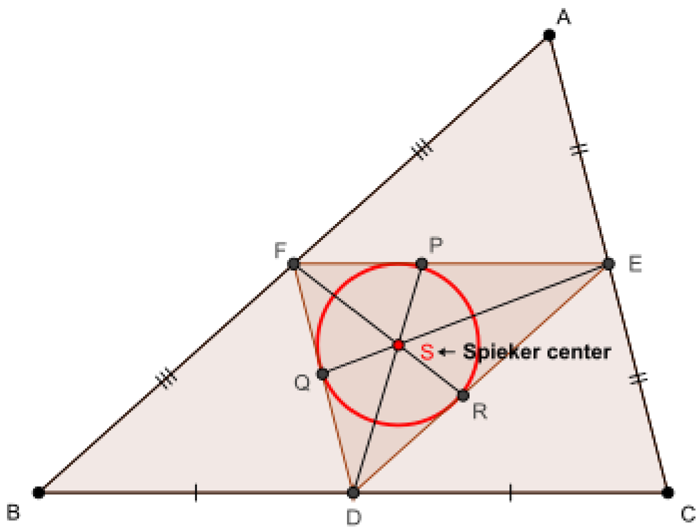
ROMANIAN MATHEMATICAL MAGAZINE

In $a$ny Δ ABC with p_a, p_b, p_c
$\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{p_a}{b+c}+\frac{p_b}{c+a}+\frac{p_c}{a+b}\le\frac{s}{4r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco
Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow(1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS,}\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$= \sum_{cyc} a^3 - abc + 2a.2bc\cos A = 2s(s^2-6Rr-3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2) \overset{(\blacksquare\blacksquare)}{=} 2s(s^2-8Rr-3r^2+8Rr\cos A)$$

$$\text{We have : } \prod_{cyc}(2s+a) = 8s^3 + 4s^2\sum_{cyc} a + 2s\sum_{cyc} ab + 4Rrs$$

$$= 8s^3 + 4s^2.2s + 2s(s^2+4Rr+r^2) + 4Rrs$$

$$\Rightarrow \prod_{cyc}(2s+a) \overset{(\blacksquare\blacksquare\blacksquare)}{=} 2s(9s^2+6Rr+r^2) \text{ and}$$

$$\sum_{cyc}(2s+b)(2s+c) = \sum_{cyc}\left(4s^2+2s(2s-a)+bc\right)$$

$$= 24s^2 - 2s(2s) + s^2 + 4Rr + r^2 \Rightarrow \sum_{cyc}(2s+b)(2s+c) \overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=} 21s^2+4Rr+r^2$$

$$(\blacksquare),(\blacksquare\blacksquare) \Rightarrow p_a = \frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A} \overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=}$$

$$\frac{2s}{2s(9s^2+6Rr+r^2)}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}.(2s+b)(2s+c) \text{ and analogs}$$

$$\Rightarrow p_a+p_b+p_c$$

$$= \frac{1}{9s^2+6Rr+r^2}.\sum_{cyc}\left(\sqrt{(s^2-8Rr-3r^2+8Rr\cos A)(2s+b)(2s+c)}.\sqrt{(2s+b)(2s+c)}\right)$$

$$\overset{CBS}{\le} \frac{1}{9s^2+6Rr+r^2}.\sqrt{\sum_{cyc}(s^2-8Rr-3r^2+8Rr\cos A)(2s+b)(2s+c)}.\sqrt{\sum_{cyc}(2s+b)(2s+c)}$$

$$\overset{\text{via }(\blacksquare\blacksquare\blacksquare\blacksquare)}{=} \frac{1}{9s^2+6Rr+r^2}.\sqrt{\begin{matrix}(s^2-8Rr-3r^2)(21s^2+4Rr+r^2)+\\ 8Rr\sum_{cyc}\left((8s^2-2sa+bc)\cos A\right)\end{matrix}}.\sqrt{21s^2+4Rr+r^2}$$

$$= \frac{1}{9s^2+6Rr+r^2}.\sqrt{\begin{matrix}(s^2-8Rr-3r^2)(21s^2+4Rr+r^2)\\ +8Rr\left(8s^2.\frac{R+r}{R} - 2s.\frac{2rs}{R} + \sum_{cyc}\left(bc.\frac{b^2+c^2-a^2}{2bc}\right)\right)\end{matrix}}.\sqrt{21s^2+4Rr+r^2}$$

$$\left(\because \sum_{cyc} a\cos A = \frac{2rs}{R}\right)$$

$$= \frac{1}{9s^2+6Rr+r^2}.\sqrt{\begin{matrix}(s^2-8Rr-3r^2)(21s^2+4Rr+r^2)\\ +8r\left(8(R+r)s^2-4rs^2+R(s^2-4Rr-r^2)\right)\end{matrix}}.\sqrt{21s^2+4Rr+r^2}$$

$$\Rightarrow (p_a+p_b+p_c)^2 \le$$

$$\frac{(21s^2+4Rr+r^2)\left(21s^4-(92Rr+30r^2)s^2-r^2(64R^2+28Rr+3r^2)\right)}{(9s^2+6Rr+r^2)^2}\overset{?}{\le}\frac{(14R-r)^2}{9}$$

$$\Leftrightarrow \boxed{\begin{gathered}3969s^6-(15876R^2+14364Rr+5562)s^4\\ -rs^2(21168R^3+15912R^2r+6804Rr^2+855r^3)\\ -r^2(7056R^4+3648R^3r+1480R^2r^2+344Rr^3+28r^4)\underset{(\bullet)}{\overset{?}{\le}}0\end{gathered}}$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\ge 0 \text{ and } s^2-(m+n)\le 0, \text{where}$$
$$m=2R^2+10Rr-r^2 \text{ and } n=2(R-2r).\sqrt{R^2-2Rr}$$
$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(\blacklozenge)}{\le}0$$
$$\therefore 3969s^2\left(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3\right)\le 0\Rightarrow \text{in order}$$
$$\text{to prove } (\bullet), \text{it suffices to prove : LHS of } (\bullet)\le$$
$$3969s^2\left(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3\right)$$

$$\Leftrightarrow \boxed{\begin{gathered}(16254R-3375r)s^4-s^2(68796R^3+51606R^2r+13608Rr^2+1206r^2)\\ -r(1764R^4+912R^3r+370R^2r^2+86Rr^3+7r^4)\underset{(\bullet\bullet)}{\overset{?}{\le}}0\end{gathered}}\ \textit{and}$$

$$\because (16254R-3375r)\left(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3\right)\overset{\text{via }(\blacklozenge)}{\le}0$$
$$\therefore \text{in order to prove } (\bullet\bullet), \text{it suffices to prove : LHS of } (\bullet\bullet)\le$$
$$(16254R-3375r)\left(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3\right)$$

$$\Leftrightarrow \boxed{\begin{gathered}(1890R^3-129987R^2r+56808Rr^2-2772r^3)s^2\\ +r(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4)\overset{(\bullet\bullet\bullet)}{\ge}0\end{gathered}}$$

$\boxed{\textit{Case 1}}$ $1890R^3-129987R^2r+56808Rr^2-2772r^3\ge 0$ *and* then : LHS of $(\bullet\bullet\bullet)$

$$\ge r(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4)>0$$
$$\Rightarrow (\bullet\bullet\bullet) \text{ is true}$$

$\boxed{\textit{Case 2}}$ $1890R^3-129987R^2r+56808Rr^2-2772r^3<0$ *and* then : LHS of $(\bullet\bullet\bullet)$

$$=-\left(-(1890R^3-129987R^2r+56808Rr^2-2772r^3)\right)s^2$$
$$+r(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4)$$
$$\overset{\text{Gerretsen}}{\ge}-\left(-(1890R^3-129987R^2r+56808Rr^2-2772r^3)\right)(4R^2+4Rr+3r^2)$$
$$+r(521010R^4+282552R^3r+16709R^2r^2-12080Rr^3-1684r^4)\overset{?}{\ge}0$$
$$\Leftrightarrow 3780t^5+1287t^4-8439t^3-85538t^2+65924t-3704\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(3780t^4+11871t^3+3713t^2+17782t(t-2)+2500)\overset{?}{\ge}0\to \text{true}$$
$$\because t\overset{\text{Euler}}{\ge}2\Rightarrow (\bullet\bullet\bullet) \text{ is true} \therefore \text{combining both cases}, (\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)$$
$$\text{is true } \forall\,\Delta\,ABC\Rightarrow (p_a+p_b+p_c)^2\le\frac{(14R-r)^2}{9}$$

$$\therefore \boxed{p_a + p_b + p_c \le \frac{14R - r}{3}} \therefore \sum_{cyc} \frac{p_a}{b + c}$$

$$= \frac{1}{2s(s^2 + 2Rr + r^2)} \cdot \sum_{cyc} \left(p_a(c + a)(a + b)\right)$$

$$= \frac{1}{2s(s^2 + 2Rr + r^2)} \cdot \sum_{cyc} \left(p_a \left(\sum_{cyc} ab\right) + a^2 p_a\right)$$

$$\le \frac{1}{2s(s^2 + 2Rr + r^2)} \cdot \left(\frac{14R - r}{3} \cdot \left(\sum_{cyc} ab\right) + \sum_{cyc} a^2 p_a\right)$$

$$\overset{\text{Chebyshev}}{\le} \frac{1}{2s(s^2 + 2Rr + r^2)} \cdot \left(\frac{14R - r}{3} \cdot \left(\sum_{cyc} ab\right) + \frac{1}{3}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} p_a\right)\right)$$

$$\left(\begin{array}{c} \because p_a = \dfrac{2s}{2s + a} \cdot \sqrt{s^2 - 3r^2 - 16Rr \sin^2 \dfrac{A}{2}} \Rightarrow \text{in order to prove} : p_a \le p_b \text{ for } a \ge b, \\ \text{it suffices to prove} : \dfrac{1}{2s + a} \le \dfrac{1}{2s + b} \text{ and} \\ s^2 - 3r^2 - 16Rr \sin^2 \dfrac{A}{2} \le s^2 - 3r^2 - 16Rr \sin^2 \dfrac{B}{2}, \text{both of which are true} \\ \therefore \text{WLOG assuming } a \ge b \ge c \Rightarrow p_a \le p_b \le p_c \text{ and } a^2 \ge b^2 \ge c^2 \end{array}\right)$$

$$\le \frac{1}{2s(s^2 + 2Rr + r^2)} \cdot \left(\frac{14R - r}{3} \cdot \left(\sum_{cyc} ab\right) + \frac{1}{3}\left(\sum_{cyc} a^2\right)\left(\frac{14R - r}{3}\right)\right)$$

$$= \frac{\frac{14R - r}{9}}{2s(s^2 + 2Rr + r^2)} \left(\left(2\sum_{cyc} ab + \sum_{cyc} a^2\right) + \sum_{cyc} ab\right)$$

$$= \frac{(14R - r)(4s^2 + s^2 + 4Rr + r^2)}{18s(s^2 + 2Rr + r^2)} \overset{?}{\le} \frac{s}{4r}$$

$$\Leftrightarrow 9s^4 - (122Rr - 19r^2)s^2 - r^2(112R^2 + 20Rr - 2r^2) \underset{(\bullet\bullet\bullet\bullet)}{\overset{?}{\ge}} 0$$

$$\text{Now, LHS of } (\bullet\bullet\bullet\bullet) \overset{\text{Gerretsen}}{\ge} (144Rr - 45r^2)s^2 - (122Rr - 19r^2)s^2$$

$$-r^2(112R^2 + 20Rr - 2r^2) = (22Rr - 26r^2)s^2 - r^2(112R^2 + 20Rr - 2r^2)$$

$$\overset{\text{Gerretsen}}{\ge} (22Rr - 26r^2)(16Rr - 5r^2) - r^2(112R^2 + 20Rr - 2r^2) \overset{?}{\ge} 0$$

$$\Leftrightarrow 40R^2 - 91Rr + 22r^2 \overset{?}{\ge} 0 \Leftrightarrow (40R - 11r)(R - 2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\Rightarrow (\bullet\bullet\bullet\bullet) \text{ is true} \therefore \frac{p_a}{b + c} + \frac{p_b}{c + a} + \frac{p_c}{a + b} \le \frac{s}{4r}$$

$\forall \Delta ABC$, " = " iff ΔABC is equilateral (QED)