

ROMANIAN MATHEMATICAL MAGAZINE

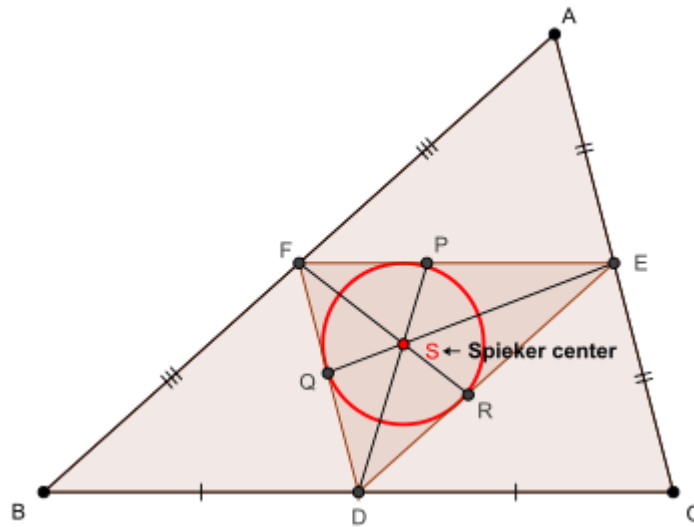
In any ΔABC with

$p_a \rightarrow$ Spieker cevian, the following relationship holds :

$$p_a \leq \frac{(b+c)^2}{16r}$$

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Let AS produced meet BC at X and $m(\angle BAX) = \alpha$ and $m(\angle CAX) = \beta$ (say)
and inradius of $\Delta DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4}\right) \left(\frac{b^2}{4}\right) - \sum \frac{a^4}{16} = \frac{1}{16} (2 \sum a^2 b^2 - \sum a^4) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\begin{aligned} \because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\angle AFS) &= B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2} \\ &= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\angle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on ΔAFS and ΔAES , we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4\sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A-B}{2} \\ &= \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A-C}{2} \end{aligned}$$

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$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2} \left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2} \right)$$

$$= Rr \left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2} \right)$$

$$= Rr \left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2 \left(1 - 2\sin^2\frac{A}{2} \right) \right)$$

$$= 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right)$$

$$= \frac{Rr}{8Rrs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a \cdot 2bcc\cos A}{8s} = \frac{bc \left((2s-a)\sin^2\frac{A}{2} - a \left(1 - 2\sin^2\frac{A}{2} \right) \right)}{2s}$$

$$= \frac{bc \left((2s+a)\sin^2\frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow - \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2}$$

$$\stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right)$$

$$= \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{4s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \stackrel{(ii)}{\Rightarrow} 2AS^2 = \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \triangle AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{4s}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b\sin\beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS}$$

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$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2} p_a c \sin \alpha + \frac{1}{2} p_a b \sin \beta = rs$$

$$\text{via (***) and (***)} \Rightarrow \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS$$

$$\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s}$$

$$\therefore p_a^2 \stackrel{(\blacksquare)}{=} \frac{2s}{(2s+a)^2} \left(b^3 + c^3 - abc + a(4m_a^2) \right)$$

$$\text{Now, } b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)$$

$$= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2)$$

$$= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2)$$

$$= (2s+a)(b^2 - bc + c^2) + a \left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2 \right)$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)((z+x) + (x+y) - 2(y+z))}{4}$$

$$- \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore b^3 + c^3 - abc + a(4m_a^2) \stackrel{(\bullet\bullet)}{=} (2s+a) \left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2} \left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2} \right)$$

$$= s(s-a) + (b-c)^2 \left(\left(\frac{s}{2s+a} \right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4} \right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left(\frac{s}{2s+a} + \frac{1}{2} \right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4} \left(\frac{(4s+a)^2}{(2s+a)^2} - 1 \right)$$

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$$\Rightarrow p_a^2 \stackrel{(\dots)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Now, $p_a \leq \frac{(b+c)^2}{16r} \Leftrightarrow p_a \cdot \frac{2rs}{a} \leq \frac{(b+c)^2}{8} \cdot \frac{s}{a} \Leftrightarrow p_a h_a \leq \frac{s(b+c)^2}{8a} \Leftrightarrow$

$$p_a^2 h_a^2 \leq \frac{s^2(b+c)^4}{64a^2} \stackrel{\text{via } (\dots)}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right) \left(s(s-a) - \frac{s(s-a)(b-c)^2}{a^2} \right) \leq \frac{s^2(b+c)^4}{64a^2}$$

$$\Leftrightarrow s^2(s-a)^2 - \frac{s^2(s-a)^2(b-c)^2}{a^2} + \frac{s(3s+a) \cdot s(s-a)(b-c)^2}{(2s+a)^2} - \frac{s(3s+a) \cdot s(s-a)(b-c)^4}{a^2(2s+a)^2} \leq \frac{s^2(b+c)^4}{64a^2}$$

$$\Leftrightarrow \frac{(2s-a)^4}{64a^2} - (s-a)^2 + (s-a)(b-c)^2 \left(\frac{s-a}{a^2} - \frac{3s+a}{(2s+a)^2} \right) + \frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \geq 0$$

$$\Leftrightarrow \frac{\left((2s-a)^2 - 8a(s-a) \right) \left((2s-a)^2 + 8a(s-a) \right)}{64a^2} + \frac{(s-a)(b-c)^2 \left((s-a)(2s+a)^2 - a^2(3s+a) \right)}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^4}{a^2(2s+a)^2} \geq 0$$

$$\Leftrightarrow \frac{(2s-3a)^2(4s^2+4sa-7a^2)}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^4 \spadesuit}{a^2(2s+a)^2} \geq 0$$

Case 1 $(2s-3a)^2 \geq (b-c)^2$ and then : LHS of $(\spadesuit) \geq \frac{(4s^2+4sa-7a^2)(b-c)^2}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} \stackrel{?}{\geq} 0$

$$\Leftrightarrow \frac{4s^2+4sa-7a^2}{64} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)}{(2s+a)^2} \stackrel{?}{\geq} 0 \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 272s^4 - 224s^3a - 392s^2a^2 + 232sa^3 + 121a^4 \stackrel{?}{\geq} 0 \text{ and}$$

$\because 272(s-a)^4 + 864a(s-a)^3 > 0 \therefore$ in order to prove (\blacksquare) , it suffices to prove : LHS of $(\blacksquare) > 272(s-a)^4 + 864a(s-a)^3$

$$\Leftrightarrow 568t^2 - 1272t + 713 > 0 \quad \left(t = \frac{s}{a} \right), \text{ which is true } \because \text{discriminant}$$

$$= 1272^2 - 4(568)(713) = -1952 < 0 \Rightarrow (\blacksquare) \Rightarrow (\spadesuit) \text{ is true}$$

Case 2 $(b-c)^2 \geq (2s-3a)^2$ and then : LHS of $(\spadesuit) \geq \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{2(s^2-a^2)(2s^2-2sa-a^2)(b-c)^2}{a^2(2s+a)^2} + \frac{(3s+a)(s-a)(b-c)^2(2s-3a)^2}{a^2(2s+a)^2} = \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2}$

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$$\begin{aligned}
 & + \frac{(s-a)(b-c)^2}{a^2(2s+a)^2} \cdot (2(s+a)(2s^2-2sa-a^2) + (3s+a)(2s-3a)^2) \\
 = & \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2} \cdot (16s^2-16sa-7a^2)
 \end{aligned}$$

which is definitely ≥ 0 if : $16s^2 - 16sa - 7a^2 \geq 0$ and so, we now consider :

$$\begin{aligned}
 & 16s^2 - 16sa - 7a^2 < 0 \text{ and } \because (b-c)^2 < a^2 \therefore \text{LHS of } (\blacklozenge) \geq \\
 & \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(b-c)^2}{a^2(2s+a)^2} \cdot (16s^2-16sa-7a^2) \\
 & > \frac{(4s^2+4sa-7a^2)(2s-3a)^2}{64a^2} + \frac{(s-a)^2(16s^2-16sa-7a^2)}{(2s+a)^2} > 0 \Leftrightarrow
 \end{aligned}$$

$$64s^6 - 64s^5a + 752s^4a^2 - 2784s^3a^3 + 2812s^2a^4 - 260sa^5 - 511a^6 \stackrel{?}{\underset{(\blacksquare\blacksquare)}}{>} 0 \text{ and}$$

$$\because (4s-5a)^6 + 104a(s-a)(4s-5a)^4 + 437a^3(4s-5a)^4 > 0$$

\therefore in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :

$$64 \cdot \text{LHS of } (\blacksquare\blacksquare) > (4s-5a)^6 + 104a(s-a)(4s-5a)^4 + 437a^3(4s-5a)^4$$

$$\Leftrightarrow 79232t^3 - 280616t^2 + 329680t - 128227 > 0$$

$$\Leftrightarrow (t-1)(79232t^2 - 201384t + 128296) + 69 > 0 \rightarrow \text{true } \because t = \frac{s}{a} > 1 \text{ and}$$

\therefore discriminant of $(79232t^2 - 201384t + 128296)$

$$= 201384^2 - 4(79232)(128296) = -105079232$$

$$\Rightarrow 79232t^2 - 201384t + 128296 > 0 \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacklozenge) \text{ is true}$$

\therefore combining both cases, (\blacklozenge) is true $\forall \Delta ABC \therefore p_a \leq \frac{(b+c)^2}{16r}$

$\forall \Delta ABC, '' = ''$ iff $2s - 3a = 0$ and $b = c \Rightarrow '' = ''$ iff ΔABC is equilateral (QED)