ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with p_a, n_a, g_a
$\rightarrow$ Spieker cevian, Nagel cevian, Gergonne cevian,
the following relationship holds :

$$w_a + 2p_a \le 4m_a - w_a \le 3p_a \le 2m_a + n_a \le g_a + 2n_a$$

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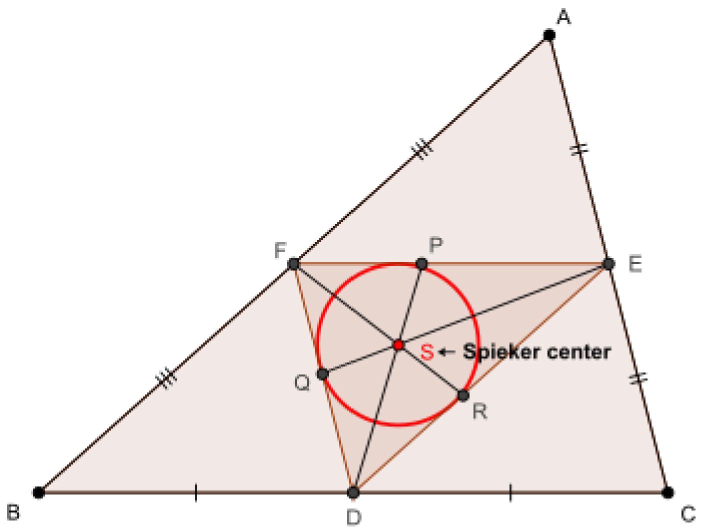

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\geq}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow\frac{s(s-a)\left(144s^3-52s^2a-16sa^2+5a^3\right)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{s(s-a)\left((s-a)\left(144s^2+92sa+76a^2\right)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)\left(1296s^3+1820s^2a+1736sa^2+1700a^3\right)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq}0\rightarrow \text{true (strict inequality)}\ \therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\rightarrow(3)$$

$$\text{Now, }(2m_a+n_a)^2-9p_a^2=4m_a^2+n_a^2+4m_an_a-9p_a^2\overset{\text{via }(3)}{\geq}$$

$$4s(s-a)+(b-c)^2+s(s-a)+\frac{s(b-c)^2}{a}+4p_a^2+\frac{2(b-c)^2}{9}-9p_a^2$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{=}5s(s-a)+\frac{11(b-c)^2}{9}+\frac{s(b-c)^2}{a}-5s(s-a)-\frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$=\left(\frac{11}{9}+\frac{s}{a}-\frac{5s(3s+a)}{(2s+a)^2}\right).(b-c)^2=\frac{36s^3-55s^2a+8sa^2+11a^3}{9a(2s+a)^2}.(b-c)^2$$

$$=\frac{(s-a)\left((s-a)(36s+17a)+6a^2\right)}{9a(2s+a)^2}.(b-c)^2\geq 0\ \therefore (2m_a+n_a)^2\geq 9p_a^2$$

$$\Rightarrow 2m_a+n_a\geq 3p_a\,\text{and analogs}\rightarrow(4)$$

$$\text{Also, }p_a^2-m_a^2\overset{\text{via }(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)-m_a^2$$

$$=\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc\right)-\left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$=\frac{4(a+b+c)\left(b^3+c^3-abc\right)-\left(2b^2+2c^2-a^2\right)(b+c)^2}{4(2s+a)^2}$$

$$=\frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2\left(b^2-c^2\right)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(\left(a^2+2a(b+c)+(b+c)^2\right)+\left((b+c)^2+2a(b+c)+a^2\right)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left(2(a+b+c)^2-a^2\right) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 \overset{(\blacksquare)}{=} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \geq w_a \Rightarrow w_a \leq p_a$$

$$\therefore \text{in order to prove : } \frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a}, \text{it suffices to prove : }$$

$$p_a^2 - m_a^2 \leq m_a^2 - w_a^2$$

$$\overset{\text{via }(\blacksquare)}{\Leftrightarrow} \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \leq s(s-a) + \frac{(b-c)^2}{4} - \left(s(s-a) - \frac{s(s-a)(b-c)^2}{(b+c)^2}\right)$$

$$= \frac{(b-c)^2}{4}\left(1+\frac{4s(s-a)}{(2s-a)^2}\right) = \frac{(b-c)^2}{4}\cdot\frac{(2s-a)^2+4s(s-a)}{(2s-a)^2}$$

$$\Leftrightarrow \left((2s-a)^2+4s(s-a)\right)(2s+a)^2 \geq (8s^2-a^2)(2s-a)^2$$

$$\Leftrightarrow 16s^3 - 12s^2a - 4sa^2 + a^3 \geq 0 \Leftrightarrow (s-a)(16s^2+4sa) + a^3 \geq 0$$

$$\rightarrow \text{true (strict) since } s > a$$

$$\therefore \frac{p_a^2-m_a^2}{p_a+m_a} \leq \frac{m_a^2-w_a^2}{m_a+w_a} \Rightarrow \boxed{p_a + w_a \leq 2m_a} \text{ and analogs} \rightarrow (5)$$

$$\text{Also, } p_a^2 - m_a^2 = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \overset{s>a}{\geq} 0 \Rightarrow p_a \geq m_a \Rightarrow p_a w_a \geq m_a w_a$$

$$\overset{\text{Lascu}+\text{A-G}}{\geq} s(s-a) \rightarrow (6)$$

$$\text{Now, } (3p_a + w_a)^2 = 9p_a^2 + w_a^2 + 6p_a w_a \overset{\text{via }(\bullet\bullet\bullet)\text{ and }(6)}{\geq}$$

$$9s(s-a) + \frac{9s(3s+a)(b-c)^2}{(2s+a)^2} + s(s-a) - \frac{s(s-a)(b-c)^2}{(2s-a)^2} + 6s(s-a)$$

$$\overset{?}{\geq} 16m_a^2 = 16s(s-a) + 4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)(b-c)^2}{(2s-a)^2} + 4(b-c)^2$$

$$\Leftrightarrow \frac{9s(3s+a)}{(2s+a)^2} \overset{?}{\geq} \frac{s(s-a)+4(2s-a)^2}{(2s-a)^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 20t^4 - 36t^3 + 13t^2 + 5t - 2 \overset{?}{\geq} 0 \quad \left(t = \frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\left((t-1)(20t^2+4t+1)+3\right) \overset{?}{\geq} 0 \rightarrow \text{true (strict)} \because t > 1$$

$$\therefore \boxed{3p_a + w_a \geq 4m_a} \text{ and analogs} \rightarrow (7)$$

$$\text{Finally, } an_a^2 . ag_a^2 \geq a^2 s^2 (s-a)^2 \Leftrightarrow$$

$$\left(b^2(s-c)+c^2(s-b)-a(s-b)(s-c)\right)\begin{pmatrix} b^2(s-b)+c^2(s-c) \\ -a(s-b)(s-c)\end{pmatrix} \overset{(a)}{\geq} a^2s^2(s-a)^2$$

Let $s-a = x, s-b = y$ *and* $s-c = z \therefore s = x+y+z \Rightarrow a = y+z, b = z+x$ *and* $c = x+y$ *and* via such substitutions, $(a) \Leftrightarrow$

$$\left(z(z+x)^2 + y(x+y)^2 - yz(y+z)\right)\left(y(z+x)^2 + z(x+y)^2 - yz(y+z)\right)$$

$$\geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0$$

$$\rightarrow \text{true} \Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \overset{(b)}{\geq} s(s-a)$$

Also, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$ and $b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$ and adding these two, we get :

$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$
$$= 2a\left(n_a^2 + g_a^2\right) + a(a+b-c)(c+a-b)$$
$$\Rightarrow 2(b^2+c^2) = 2\left(n_a^2+g_a^2\right) + a^2 - (b-c)^2$$
$$\Rightarrow 2\left(b^2+c^2\right) - a^2 + (b-c)^2 = 2\left(n_a^2+g_a^2\right) \Rightarrow 4m_a^2 + (b-c)^2 = 2\left(n_a^2+g_a^2\right)$$
$$\Rightarrow 2(b-c)^2 + 4s(s-a) = 2\left(n_a^2+g_a^2\right) \Rightarrow n_a^2 + g_a^2 = (b-c)^2 + 2s(s-a)$$
$$\Rightarrow n_a^2 + g_a^2 + 2n_a g_a \overset{\text{via (b)}}{\geq} (b-c)^2 + 4s(s-a) \Rightarrow (n_a+g_a)^2 \geq 4m_a^2$$
$$\Rightarrow \boxed{n_a + g_a \geq 2m_a} \text{ and analogs} \rightarrow (8)$$

Now, $w_a + 2p_a \leq 4m_a - w_a$ is equivalent to (5) and then :
$4m_a - w_a \leq 3p_a$ is equivalent to (7) and also,
$3p_a \leq 2m_a + n_a$ is equivalent to (4) and finally,
$2m_a + n_a \leq g_a + 2n_a$ is equivalent to (8)

$$\therefore w_a + 2p_a \leq 4m_a - w_a \leq 3p_a \leq 2m_a + n_a \leq g_a + 2n_a \ \forall \ \Delta ABC \ (\text{QED})$$