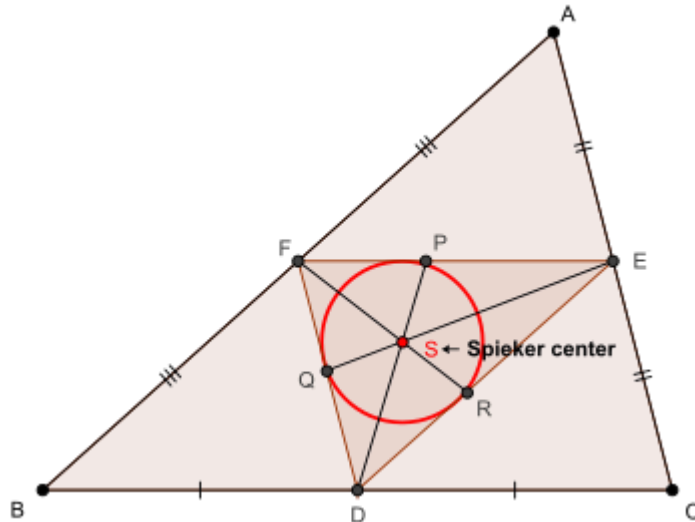**In *any* △ ABC with**

$p_a \rightarrow$ **Spieker cevi*an*, the fo*llowing* re*lationship* ho*lds* :**

$$\frac{p_a}{h_a} \leq \frac{2R}{3r} - \frac{1}{3}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

Let AS produced meet BC *at* X *and* m($\sphericalangle$BAX) = α *and* m($\sphericalangle$CAX) = β (*say*)
***and* in*radius of* △ DEF = r′(*say*)**

$$\text{Now, } 16[\text{DEF}]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [\text{DEF}] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$\because$ **Spieker center is incenter of** △ DEF, $\therefore$ m($\sphericalangle$AFS) = $B + \dfrac{C}{2} = \dfrac{2B+C}{2} = \dfrac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ } and \text{ m}(\sphericalangle\text{AES}) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Vi*a* (1), (2) *and* using cosine *law* on △ AFS *and* △ AES, we *arrive at* :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,} \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1-2\sin^2\frac{B}{2} + 1-2\sin^2\frac{C}{2} - 2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a\cdot 2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2 s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$\text{(i), (*), (**)} \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, [BAX] + [BAX] = [ABC]} \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{via\ (***)\ and\ (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{via\ (ii)}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\cdot)}{=} \frac{2s}{(2s+a)^2}\left(b^3 + c^3 - abc + a(4m_a^2)\right)}$$

$$\text{Now, } b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)$$

$$= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2)$$

$$= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2)$$

$$= (2s+a)(b^2 - bc + c^2) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right)$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \cdot$$

$$\frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)\big((z+x) + (x+y) - 2(y+z)\big)}{4}$$

$$- \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2}\right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3 + c^3 - abc + a(4m_a^2) \overset{(\cdot\cdot)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2}\right) + \frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s-a) + (b-c)^2\left(\left(\frac{s}{2s+a}\right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\cdot\cdot\cdot)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{via\ (\cdot\cdot\cdot)}{\Longleftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+ \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9} \cdot \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a)\cdot\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a)\cdot\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ \left(\because (b-c)^2\geq 0\right)$$

$$\Leftrightarrow \frac{s(s-a)\left(144s^3-52s^2a-16sa^2+5a^3\right)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}\cdot(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+1701a^4\right)}{324a(2s+a)^4}\cdot(b-c)^2$$

$$\overset{?}{\geq}0\to\textbf{true (strict inequality)}\ \therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\to(3)$$

$$\textbf{Now, }(2m_a+n_a)^2-9p_a^2=4m_a^2+n_a^2+4m_an_a-9p_a^2\overset{via\,(3)}{\geq}$$

$$4s(s-a)+(b-c)^2+s(s-a)+\frac{s(b-c)^2}{a}+4p_a^2+\frac{2(b-c)^2}{9}-9p_a^2$$

$$\overset{via\,(\cdots)}{=}5s(s-a)+\frac{11(b-c)^2}{9}+\frac{s(b-c)^2}{a}-5s(s-a)-\frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$=\left(\frac{11}{9}+\frac{s}{a}-\frac{5s(3s+a)}{(2s+a)^2}\right)\cdot(b-c)^2=\frac{36s^3-55s^2a+8sa^2+11a^3}{9a(2s+a)^2}\cdot(b-c)^2$$

$$=\frac{(s-a)\left((s-a)(36s+17a)+6a^2\right)}{9a(2s+a)^2}\cdot(b-c)^2\geq 0\ \therefore(2m_a+n_a)^2\geq 9p_a^2$$

$$\Rightarrow 2m_a+n_a\geq 3p_a\ and\ analogs\to(4)$$

$$\textbf{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$

$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=$$

$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\ \therefore n_a^2=s^2-2r_ah_a$$

$$\therefore a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\leq}4(R-r)^2s^2\Leftrightarrow$$

$$(4R^2\sin^2A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\leq}4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\geq}0\Leftrightarrow R^2\cos^2A-2Rr\cos A+r^2\overset{?}{\geq}0$$

$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\geq}0\to\textbf{true}\ \therefore an_a\leq 2Rs-2rs\Rightarrow\frac{n_a}{h_a}\leq\frac{2Rs}{a\left(\frac{2rs}{a}\right)}-\frac{2rs}{a\left(\frac{2rs}{a}\right)}$$

$$\Rightarrow \frac{n_a}{h_a} \le \frac{R}{r} - 1 \; and \; analogs \to (5) \; \therefore \; (4) \Rightarrow \frac{3p_a}{h_a} \le \frac{2m_a}{h_a} + \frac{n_a}{h_a} \overset{\substack{via\,(5)\\ and \\ Panaitopol}}{\le} \frac{R}{r} + \frac{R}{r} - 1$$

$$\Rightarrow \frac{p_a}{h_a} \le \frac{2R}{3r} - \frac{1}{3} \; \forall \, \Delta \, \mathbf{ABC} \; \mathbf{(QED)}$$