ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with:

$p_a, p_b, p_c \to$ Spieker cevians, the following relationship holds :

$$\frac{p_a}{m_a}+\frac{p_b}{m_b}+\frac{p_c}{m_c}\le\frac{R}{2r}+2$$

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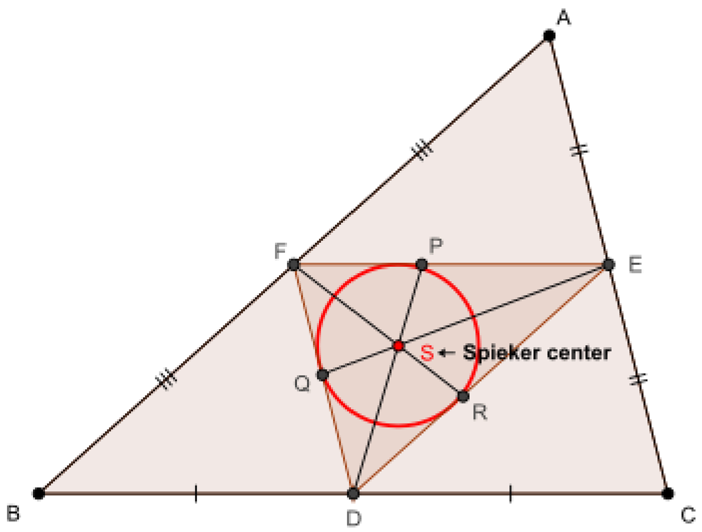

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS},\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES}, b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } m_a n_a \overset{?}{\geq} p_a^2+\frac{(b-c)^2}{18} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$
$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\geq}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\left(\because (b-c)^2\geq 0\right)$$

$$\Leftrightarrow \frac{s(s-a)\left(144s^3-52s^2a-16sa^2+5a^3\right)}{36a(2s+a)^2}$$

$$+\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)\left(144s^2+92sa+76a^2\right)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left((s-a)\left(1296s^3+1820s^2a+1736sa^2+1700a^3\right)+1701a^4\right)}{324a(2s+a)^4}.(b-c)^2$$

$$\overset{?}{\geq}0\to \text{true (strict inequality)}\ \therefore m_an_a\geq p_a^2+\frac{(b-c)^2}{18}\to(3)$$

$$\text{Now, }(2m_a+n_a)^2-9p_a^2=4m_a^2+n_a^2+4m_an_a-9p_a^2\overset{\text{via }(3)}{\geq}$$

$$4s(s-a)+(b-c)^2+s(s-a)+\frac{s(b-c)^2}{a}+4p_a^2+\frac{2(b-c)^2}{9}-9p_a^2$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{=}5s(s-a)+\frac{11(b-c)^2}{9}+\frac{s(b-c)^2}{a}-5s(s-a)-\frac{5s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$=\left(\frac{11}{9}+\frac{s}{a}-\frac{5s(3s+a)}{(2s+a)^2}\right).(b-c)^2=\frac{36s^3-55s^2a+8sa^2+11a^3}{9a(2s+a)^2}.(b-c)^2$$

$$=\frac{(s-a)\left((s-a)(36s+17a)+6a^2\right)}{9a(2s+a)^2}.(b-c)^2\geq 0\ \therefore (2m_a+n_a)^2\geq 9p_a^2$$

$$\Rightarrow 2m_a+n_a\geq 3p_a \text{ and analogs}\ \therefore \frac{p_a}{m_a}+\frac{p_b}{m_b}+\frac{p_c}{m_c}\leq\sum_{\text{cyc}}\frac{2m_a+n_a}{3m_a}$$

$$=2+\sum_{\text{cyc}}\frac{n_a}{3m_a}\overset{?}{\leq}\frac{R}{2r}+2\Leftrightarrow \boxed{\sum_{\text{cyc}}\frac{n_a}{m_a}\underset{(\blacksquare)}{\overset{?}{\leq}}\frac{3R}{2r}}$$

$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$

$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s\left(b^2+c^2\right)-2sbc$$

$$=an_a^2+a\left(as-s^2\right)\Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2\Rightarrow an_a^2=$$

$$as^2+s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$=as^2-\frac{4\Delta^2}{s-a}=as^2-2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=as^2-2ah_ar_a\ \therefore n_a^2=s^2-2r_ah_a$$

$$=as^2-2ah_ar_a\ \therefore n_a^2=s^2-2r_ah_a=s^2-\frac{4rs^2\tan\frac{A}{2}}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}=s^2-\frac{rs^2}{R}\sec^2\frac{A}{2}$$

$$\Rightarrow n_a^2(s-b)(s-c)=s^2(s-b)(s-c)-\frac{rs^2}{R}.\frac{(s-b)(s-c)}{s(s-a)}.bc$$

$$= s^2(s-b)(s-c) - \frac{rs^2}{R}.\tan^2\frac{A}{2}.\frac{4Rrs}{4R\cos^2\frac{A}{2}\tan\frac{A}{2}}$$

$$= s^2(s-b)(s-c) - \frac{r^2s^3}{R}.\tan\frac{A}{2}\left(1+\tan^2\frac{A}{2}\right) \therefore n_a^2(s-b)(s-c)$$

$$= s^2(s-b)(s-c) - \frac{r^2s^2}{R}.r_a - \frac{r^2}{R}.r_a^3 \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\left(n_a^2(s-b)(s-c)\right) = s^2\sum_{cyc}(s-b)(s-c) - \frac{r^2s^2}{R}.\sum_{cyc} r_a - \frac{r^2}{R}.\sum_{cyc} r_a^3$$

$$= s^2(4Rr+r^2) - \frac{r^2s^2(4R+r)}{R} - \frac{r^2}{R}.\left((4R+r)^3 - 3\prod_{cyc}\left(4R\cos^2\frac{A}{2}\right)\right)$$

$$= s^2(4Rr+r^2) - \frac{r^2s^2(4R+r)}{R} - \frac{r^2}{R}.\left((4R+r)^3 - 3.64R^3.\frac{s^2}{16R^2}\right)$$

$$= \frac{r\left((4R^2+9Rr-r^2)s^2 - r(4R+r)^3\right)}{R}$$

$$\Rightarrow \frac{\sum_{cyc}\left(n_a^2(s-b)(s-c)\right)}{s(s-a)(s-b)(s-c)} = \frac{r\left((4R^2+9Rr-r^2)s^2 - r(4R+r)^3\right)}{Rr^2s^2}$$

$$\Rightarrow \boxed{\sum_{cyc}\frac{n_a^2}{s(s-a)} = \frac{r\left((4R^2+9Rr-r^2)s^2 - r(4R+r)^3\right)}{Rr^2s^2}} \therefore \sum_{cyc}\frac{n_a}{m_a} \overset{Lascu + A-G}{\leq}$$

$$\sum_{cyc}\frac{n_a}{\sqrt{s(s-a)}} \overset{CBS}{\leq} \sqrt{3\sum_{cyc}\frac{n_a^2}{s(s-a)}} = \sqrt{\frac{3r\left((4R^2+9Rr-r^2)s^2 - r(4R+r)^3\right)}{Rr^2s^2}} \overset{?}{\leq} \frac{3R}{2r}$$

$$\Leftrightarrow (3R^3 - 16R^2r - 36Rr^2 + 4r^3)s^2 + 4r^2(4R+r)^3 \underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}} 0$$

$\boxed{Case\ 1}$ $3R^3 - 16R^2r - 36Rr^2 + 4r^3 \geq 0$ *and then* : LHS of $(\blacksquare\blacksquare) \geq 4r^2(4R+r)^3 > 0 \Rightarrow (\blacksquare\blacksquare)$ is true (strict inequality)

$\boxed{Case\ 2}$ $3R^3 - 16R^2r - 36Rr^2 + 4r^3 < 0$ *and then* : LHS of $(\blacksquare\blacksquare) \overset{Gerretsen}{\geq}$

$$(3R^3 - 16R^2r - 36Rr^2 + 4r^3)(4R^2+4Rr+3r^2) + 4r^2(4R+r)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow 12t^5 - 52t^4 + 57t^3 + 16t^2 - 44t + 16 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)^2\left((t-2)(12t^2+20t+33)+70\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (\blacksquare\blacksquare) \text{ is true}$$

$\therefore$ combining both cases, $(\blacksquare\blacksquare) \Rightarrow (\blacksquare)$ is true $\forall \Delta ABC \therefore \frac{p_a}{m_a} + \frac{p_b}{m_b} + \frac{p_c}{m_c} \leq \frac{R}{2r} + 2$

$\forall \Delta ABC$,'' ='' iff ΔABC is equilateral (QED)