In *any* Δ ABC with

$n_a \to$ Nagel cevian, the following relationship holds :

$$n_a \geq \frac{b^2 - bc + c^2}{2R}$$

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Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$\Rightarrow s(b^2 + c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc$

$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$

$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$

$= as^2 - s(a^2 - (b-c)^2) = as(s-a) + s(b-c)^2$

$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2 \to (1)$

Now, $n_a \geq \frac{b^2 - bc + c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \geq \frac{b^2 - bc + c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \geq \left(\frac{b^2 - bc + c^2}{bc}\right)^2 - 1$

$= \frac{(b-c)^2(b^2 + c^2)}{b^2c^2} \Leftrightarrow \frac{n_a^2 - h_a^2}{h_a^2} \geq \frac{(b-c)^2(b^2 + c^2)}{b^2c^2} \overset{via\ (1)}{\Leftrightarrow}$

$s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2} \geq \frac{(b-c)^2(b^2 + c^2)}{b^2c^2} \cdot \frac{b^2c^2}{4R^2}$

$\Leftrightarrow \left(\frac{s}{a} + \frac{s(s-a)}{a^2}\right)(b-c)^2 \geq \frac{(b-c)^2(b^2 + c^2)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \geq \frac{b^2 + c^2}{4R^2} \ (\because (b-c)^2 \geq 0)$

$\Leftrightarrow 4R^2s^2 \geq a^2b^2 + c^2a^2 \to$ true (strict inequality) $\because 4R^2s^2 \overset{Goldstone}{\geq}$

$\sum_{cyc} a^2b^2 > a^2b^2 + c^2a^2 \therefore n_a \geq \frac{b^2 - bc + c^2}{2R}, '' = ''$ iff $b = c$ (QED)