ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c $\rightarrow$ Nagel's cevians, the following relationship holds:

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-3$$

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$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s(a^2-(b-c)^2)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2\rightarrow(1)$$

$$\text{Now, }\frac{n_a}{h_a}\overset{?}{\ge}\frac{b^2-bc+c^2}{bc}\Leftrightarrow\frac{n_a^2}{h_a^2}-1\overset{?}{\ge}\left(\frac{b^2-bc+c^2}{bc}\right)^2-1=\frac{(b-c)^2(b^2+c^2)}{b^2c^2}$$
$$\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\overset{\text{via }(1)}{\Leftrightarrow}$$
$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{4R^2}\Leftrightarrow\frac{s^2}{a^2}\overset{?}{\ge}\frac{b^2+c^2}{4R^2}\ (\because (b-c)^2\ge 0)$$
$$\Leftrightarrow 4R^2s^2\overset{?}{\ge}a^2b^2+c^2a^2\rightarrow\text{true (strict inequality)}\because 4R^2s^2\overset{\text{Goldstone}}{\ge}$$
$$\sum_{\text{cyc}}a^2b^2>a^2b^2+c^2a^2\ \therefore\frac{n_a}{h_a}\ge\frac{b^2-bc+c^2}{bc}\text{ and analogs}$$
$$\Rightarrow\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge\sum_{\text{cyc}}\frac{b^2-bc+c^2}{bc}=\sum_{\text{cyc}}\left(\frac{b}{c}+\frac{c}{b}\right)-3=\sum_{\text{cyc}}\frac{b+c}{a}-3$$
$$\therefore\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ge\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-3$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)