ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with p_a, p_b, p_c
$\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\le\frac{4R+r}{3r}$$

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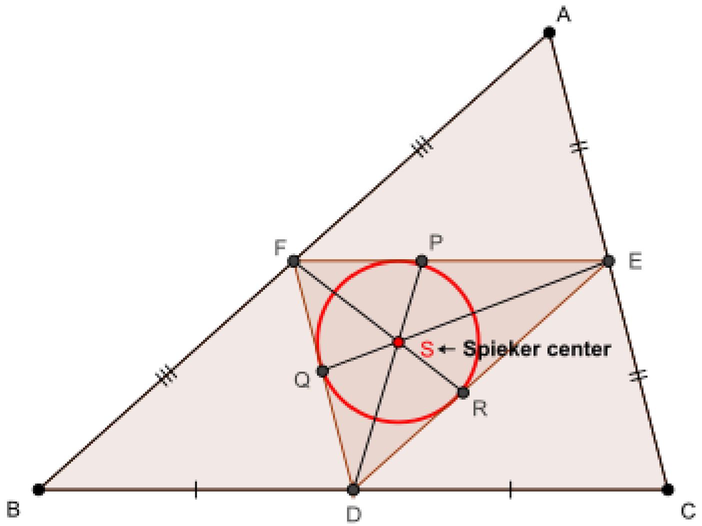

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2} \overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2) + a(b^2-bc+c^2) + a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2) + a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2) + a\left(\frac{(b+c)^2-(b-c)^2}{4} - a^2\right)$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4} - \frac{a(b-c)^2}{4} =$$

$$\binom{2s}{+a}.\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2}\right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2}\right) + \frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s-a) + (b-c)^2\left(\left(\frac{s}{2s+a}\right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2.\left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$\Rightarrow \boxed{ap_a^2 \overset{(\bullet\bullet\bullet)}{=} as(s-a) - \frac{a(b-c)^2}{4} + \frac{a(4s+a)^2}{(4s+2a)^2}.(b-c)^2}$$

$$\text{Now, } \frac{a(4s+a)^2}{(4s+2a)^2} = a.\frac{(4s+2a)^2 - 2a(4s+2a) + a^2}{(4s+2a)^2}$$

$$= a - \frac{(a+2s-2s)^2}{2s+a} + \frac{(a+2s-2s)^3}{(4s+2a)^2}$$

$$= a - (2s+a) + 4s - \frac{4s^2}{2s+a} + \frac{1}{4}\left(\frac{(2s+a)^3 - 8s^3 - 3(2s+a)(2s)a}{(2s+a)^2}\right)$$

$$= 2s - \frac{4s^2}{2s+a} + \frac{2s+a}{4} - \frac{2s^3}{(2s+a)^2} - \frac{3s(a+2s-2s)}{2(2s+a)}$$

$$= \frac{5s}{2} + \frac{a}{4} - \frac{4s^2}{2s+a} - \frac{2s^3}{(2s+a)^2} - \frac{3s}{2} + \frac{3s^2}{2s+a}$$

$$\therefore \boxed{\frac{a(4s+a)^2}{(4s+2a)^2} \overset{(\bullet\bullet\bullet\bullet)}{=} s + \frac{a}{4} - \frac{s^2(4s+a)}{(2s+a)^2}}$$

$$\therefore (\bullet\bullet\bullet), (\bullet\bullet\bullet\bullet) \Rightarrow ap_a^2 =$$

$$as(s-a) - \frac{a(b-c)^2}{4} + s(b-c)^2 + \frac{a(b-c)^2}{4}$$

$$-\frac{s^2(4s+a)}{(2s+a)^2}.(b-c)^2 \overset{a<s}{\leq} as(s-a) + s(b-c)^2 - \frac{s^2(4s+a)}{(2s+s)^2}.(b-c)^2$$

$$= as(s-a) + s(b-c)^2 - \frac{(4s+a)(b-c)^2}{9}$$

$$= as(s-a) + \frac{5s(b-c)^2}{9} - \frac{a(b-c)^2}{9} \text{ and analogs}$$

$$\therefore \sum_{cyc} ap_a^2 = s\left(2s - 2\left(s^2 - 4Rr - r^2\right)\right) + \frac{10s}{9}\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right)$$

$$-\frac{1}{9}\left(\sum_{cyc}\left(ab(2s-c)\right) - 6abc\right)$$

$$= s\left(8Rr + 2r^2\right) + \frac{10s\left(s^2 - 12Rr - 3r^2\right)}{9} - \frac{2s\left(s^2 - 14Rr + r^2\right)}{9}$$

$$= \frac{2s\left(4s^2 - 10Rr - 7r^2\right)}{9} \Rightarrow \frac{\sum_{cyc} ap_a^2}{2rs} \overset{(\blacklozenge)}{\leq} \frac{4s^2 - 10Rr - 7r^2}{9r} \text{ and } \frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \overset{CBS}{\leq}$$

$$\sqrt{\sum_{cyc}\frac{p_a^2}{h_a}}.\sqrt{\sum_{cyc}\frac{1}{h_a}} = \sqrt{\frac{\sum_{cyc} ap_a^2}{2rs}}.\sqrt{\frac{1}{r}} \overset{\text{via } (\blacklozenge)}{\leq} \sqrt{\frac{4s^2 - 10Rr - 7r^2}{9r^2}}$$

$$\overset{\text{Gerretsen}}{\leq} \sqrt{\frac{4(4R^2 + 4Rr + 3r^2) - 10Rr - 7r^2}{9r^2}}$$

$$= \sqrt{\frac{16R^2 + 6Rr + 5r^2}{9r^2}} \overset{\text{Euler}}{\leq} \sqrt{\frac{16R^2 + 6Rr + r^2 + 2Rr}{9r^2}} = \sqrt{\frac{(4R+r)^2}{9r^2}}$$

$$\therefore \frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \leq \frac{4R+r}{3r}\ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$