ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c $\rightarrow$ Nagel's cevians, the following relationship holds :

$$\frac{\sqrt{n_b n_c}}{h_a}+\frac{\sqrt{n_c n_a}}{h_b}+\frac{\sqrt{n_a n_b}}{h_c} \geq \frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc) \Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2 \Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s(a^2-(b-c)^2)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2 \rightarrow (1)$$

$$\text{Now, } n_a \overset{?}{\geq} \frac{b^2-bc+c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\geq} \frac{b^2-bc+c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2}-1 \overset{?}{\geq} \left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$
$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2} \Leftrightarrow \frac{n_a^2-h_a^2}{h_a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2} \overset{\text{via (1)}}{\Leftrightarrow}$$
$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow \left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \overset{?}{\geq} \frac{b^2+c^2}{4R^2} \ (\because (b-c)^2 \geq 0)$$
$$\Leftrightarrow 4R^2s^2 \overset{?}{\geq} a^2b^2+c^2a^2 \rightarrow \text{true (strict inequality)} \because 4R^2s^2 \overset{\text{Goldstone}}{\geq} \sum_{\text{cyc}} a^2b^2 >$$
$$a^2b^2+c^2a^2 \therefore n_a \overset{(\bullet)}{\geq} \frac{b^2-bc+c^2}{2R} \text{ and analogs}$$
$$\therefore 4R^2.n_b n_c \geq (c^2-ca+a^2)(a^2-ab+b^2)$$
$$=\left((x+y)^2-(x+y)(y+z)+(y+z)^2\right)\left((y+z)^2-(y+z)(z+x)+(z+x)^2\right)$$
$$(x=s-a, y=s-b, z=s-c)=\sum_{\text{cyc}} x^4+\sum_{\text{cyc}} x^2y^2+2yz(y^2+z^2+yz)+4x^2yz$$
$$=(y^4+z^4+2y^2z^2)+(x^4+y^2z^2+2x^2yz)+(x^2y^2+x^2z^2+2x^2yz)$$
$$+2yz(y^2+z^2) \geq (y^2+z^2)^2+(x^2+yz)^2+x^2(y+z)^2+yz(y+z)^2$$
$$\geq \frac{(y+z)^4}{4}+(x^2+yz)^2+(x^2+yz)(y+z)^2$$
$$=\frac{(y+z)^4+4(x^2+yz)^2+4(x^2+yz)(y+z)^2}{4}=\frac{\left((y+z)^2+2(x^2+yz)\right)^2}{4}$$
$$=\frac{\left(a^2+2\left((s-a)^2+(s-b)(s-c)\right)\right)^2}{4}$$

$$=\frac{\left(a^2+2(s^2-2sa+a^2+s^2-s(2s-a)+bc)\right)^2}{4}=\frac{(3a^2-a(a+b+c)+2bc)^2}{4}$$

$$\Rightarrow n_b n_c \geq \frac{(2a^2+2bc-ab-ac)^2}{16R^2}$$

$$\Rightarrow \boxed{\sqrt{n_b n_c} \overset{(\bullet\bullet)}{\geq} \frac{2a^2+2bc-ab-ac}{4R}} \text{ and analogs}$$

$$(\because 2a^2+2bc-ab-ac=(y+z)^2+2(x^2+yz)>0) \text{ and } \sum_{cyc}\frac{h_a}{r_a}=\sum_{cyc}\frac{2(s-a)}{a}$$

$$=\frac{2s}{4Rrs}.\sum_{cyc}ab-6=\frac{s^2+4Rr+r^2}{2Rr}-6 \Rightarrow \sum_{cyc}\frac{h_a}{r_a}=\frac{s^2-8Rr+r^2}{2Rr}$$

$$\therefore \frac{\sqrt{n_b n_c}}{h_a}+\frac{\sqrt{n_c n_a}}{h_b}+\frac{\sqrt{n_a n_b}}{h_c} \geq \frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}$$

$$\Leftrightarrow \boxed{\sum_{cyc}\frac{n_b n_c}{h_a^2}+2\sum_{cyc}\left(\frac{\sqrt{n_b n_c}}{h_b h_c}.n_a\right) \overset{(*)}{\geq} \frac{(s^2-8Rr+r^2)^2}{4R^2r^2}}$$

$$\text{Now, via } (\bullet), \boxed{\boxed{\sum_{cyc}\frac{n_b n_c}{h_a^2} \geq}} \sum_{cyc}\frac{a^2(c^2-ca+a^2)(a^2-ab+b^2)}{4R^2.4r^2s^2}$$

$$=\frac{\sum_{cyc}a^6-\sum_{cyc}\left(ab(\sum_{cyc}a^4-c^4)\right)+\sum_{cyc}\left(a^2b^2(\sum_{cyc}a^2-c^2)\right)+abc\sum_{cyc}a^3+3a^2b^2c^2}{16R^2r^2s^2}$$

$$\frac{-abc\left((\sum_{cyc}a)(\sum_{cyc}ab)-3abc\right)}{16R^2r^2s^2}$$

$$=\boxed{\boxed{\frac{2\left(s^6-(16Rr+9r^2)s^4+r^2s^2(96R^2+76Rr+19r^2)-3r^3(4R+r)^3\right)}{16R^2r^2s^2}}} \rightarrow \text{(i)}$$

$$\left(\begin{array}{c}\text{using } \sum_{cyc}a^6=4s^2(s^2-6Rr-3r^2)^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right), \\ \sum_{cyc}a^4=2\sum_{cyc}a^2b^2-16r^2s^2 \text{ and } \sum_{cyc}a^2b^2=(s^2+4Rr+r^2)^2-16Rrs^2\end{array}\right)$$

$$\text{Now, via } (\bullet) \text{ and } (\bullet\bullet), \boxed{\boxed{2\sum_{cyc}\left(\frac{\sqrt{n_b n_c}}{h_b h_c}.n_a\right)}} \geq$$

$$2.\sum_{cyc}\left(\frac{2a^2+2bc-ab-ac}{4R}.\frac{b^2-bc+c^2}{2R}.\frac{bc}{\frac{16R^2r^2s^2}{4R^2}}\right)$$

$$=\frac{abc\sum_{cyc}\left((2a-b-c)(b^2-bc+c^2)\right)+2\sum_{cyc}\left(b^2c^2(b^2-bc+c^2)\right)}{16R^2r^2s^2}$$

$$=\frac{2abc\left((\sum_{cyc}a)(\sum_{cyc}ab)-6abc-\left(3abc+(\sum_{cyc}a)(\sum_{cyc}a^2)-(\sum_{cyc}a)(\sum_{cyc}ab)\right)\right)}{16R^2r^2s^2}$$

$$\frac{+(\sum_{cyc}a^2)(\sum_{cyc}a^2b^2)-3a^2b^2c^2-\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)}{16R^2r^2s^2}$$

$$= \boxed{\dfrac{2\left(s^6-\left(12Rr+r^2\right)s^4+r^2s^2\left(48R^2+16Rr-5r^2\right)-3r^3(4R+r)^3\right)-32Rr^2s^2(R-2r)}{16R^2r^2s^2}}$$

$\rightarrow$ (ii) $\therefore$ (i) *and* (ii) $\Rightarrow$ LHS of $(*) \geq$

$$\frac{4\left(s^6-\left(14Rr+5r^2\right)s^4+r^2s^2\left(72R^2+46Rr+7r^2\right)-3r^3(4R+r)^3-32Rr^2s^2(R-2r)\right)}{16R^2r^2s^2} \overset{?}{\geq}$$

$$\frac{\left(s^2-8Rr+r^2\right)^2}{4R^2r^2} \Leftrightarrow (2R-7r)s^4+r^2s^2(78R+6r)-3r^2(4R+r)^3 \underset{(**)}{\overset{?}{\geq}} 0$$

Now, $(2R-7r)s^4+r^2s^2(78R+6r)=(2R-4r)s^4-3rs^4+r^2s^2(78R+6r)$

$$\overset{\text{Gerretsen}}{\geq} (2R-4r)\left(16Rr-5r^2\right)s^2-3r\left(4R^2+4Rr+3r^2\right)s^2+r^2s^2(78R+6r)$$

$$= r\left(20R^2-8Rr+17r^2\right)s^2 \overset{\text{Gerretsen}}{\geq} r\left(20R^2-8Rr+17r^2\right)\left(16Rr-5r^2\right)$$

$$\overset{?}{\geq} 3r^2(4R+r)^3 \Leftrightarrow 32t^3-93t^2+69t-22 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(17t^2+15t(t-2)+t+11\right) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \frac{\sqrt{n_b n_c}}{h_a}+\frac{\sqrt{n_c n_a}}{h_b}+\frac{\sqrt{n_a n_b}}{h_c} \geq \frac{h_a}{r_a}+\frac{h_b}{r_b}+\frac{h_c}{r_c}\ \forall \Delta ABC,$$

with equality iff Δ ABC is equilateral (QED)