ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c $\to$ Nagel's cevians, the following relationship holds :

$$\sqrt{n_a n_b}+\sqrt{n_b n_c}+\sqrt{n_c n_a}\ge\frac{a^2+b^2+c^2}{2R}$$

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$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s(a^2-(b-c)^2)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2\to(1)$$

$$\text{Now, } n_a\overset{?}{\ge}\frac{b^2-bc+c^2}{2R}\Leftrightarrow\frac{n_a}{h_a}\overset{?}{\ge}\frac{b^2-bc+c^2}{bc}\Leftrightarrow\frac{n_a^2}{h_a^2}-1\overset{?}{\ge}\left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$
$$=\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\overset{\text{via }(1)}{\Leftrightarrow}$$
$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\ge}\frac{(b-c)^2(b^2+c^2)}{4R^2}\Leftrightarrow\frac{s^2}{a^2}\overset{?}{\ge}\frac{b^2+c^2}{4R^2}\ (\because(b-c)^2\ge0)$$
$$\Leftrightarrow 4R^2s^2\overset{?}{\ge}a^2b^2+c^2a^2\to\text{true (strict inequality)}\because 4R^2s^2\overset{\text{Goldstone}}{\ge}\sum_{\text{cyc}}a^2b^2>$$
$$a^2b^2+c^2a^2\therefore n_a\overset{(\bullet)}{\ge}\frac{b^2-bc+c^2}{2R}\text{ and analogs}$$
$$\therefore 4R^2.n_bn_c\ge(c^2-ca+a^2)(a^2-ab+b^2)$$
$$=((x+y)^2-(x+y)(y+z)+(y+z)^2)((y+z)^2-(y+z)(z+x)+(z+x)^2)$$
$$(x=s-a,y=s-b,z=s-c)=\sum_{\text{cyc}}x^4+\sum_{\text{cyc}}x^2y^2+2yz(y^2+z^2+yz)+4x^2yz$$
$$=(y^4+z^4+2y^2z^2)+(x^4+y^2z^2+2x^2yz)+(x^2y^2+x^2z^2+2x^2yz)$$
$$+2yz(y^2+z^2)\ge(y^2+z^2)^2+(x^2+yz)^2+x^2(y+z)^2+yz(y+z)^2$$
$$\ge\frac{(y+z)^4}{4}+(x^2+yz)^2+(x^2+yz)(y+z)^2$$
$$=\frac{(y+z)^4+4(x^2+yz)^2+4(x^2+yz)(y+z)^2}{4}=\frac{\left((y+z)^2+2(x^2+yz)\right)^2}{4}$$
$$=\frac{\left(a^2+2\left((s-a)^2+(s-b)(s-c)\right)\right)^2}{4}$$

$$=\frac{\left(a^2+2\left(s^2-2sa+a^2+s^2-s(2s-a)+bc\right)\right)^2}{4}=\frac{\left(3a^2-a(a+b+c)+2bc\right)^2}{4}$$

$$\Rightarrow n_b n_c \geq \frac{\left(2a^2+2bc-ab-ac\right)^2}{16R^2}$$

$$\Rightarrow \boxed{\sqrt{n_b n_c} \overset{(\bullet\bullet)}{\geq} \frac{2a^2+2bc-ab-ac}{4R}} \text{ and analogs}$$

$$\left(\because 2a^2+2bc-ab-ac=(y+z)^2+2\left(x^2+yz\right)>0\right)$$

$$\therefore \sqrt{n_a n_b}+\sqrt{n_b n_c}+\sqrt{n_c n_a} \geq \sum_{cyc}\frac{2a^2+2bc-ab-ac}{4R}=\frac{a^2+b^2+c^2}{2R}\ \forall\ \Delta\ ABC,$$

with equality iff Δ ABC is equilateral (QED)