ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with n_a, n_b, n_c
$\rightarrow$ Nagel's cevians, the following relationship holds :

$$\frac{\sqrt{m_a n_a}}{h_a}+\frac{\sqrt{m_b n_b}}{h_b}+\frac{\sqrt{m_c n_c}}{h_c}\ge\frac{\sqrt{2}}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-3\sqrt{2}$$

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$$\text{Stewart's theorem}\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right)\Rightarrow s\left(b^2+c^2\right)-2sbc$$
$$=an_a^2+a\left(as-s^2\right)\Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-s\left(a^2-(b-c)^2\right)=as(s-a)+s(b-c)^2$$
$$\Rightarrow n_a^2=s(s-a)+\frac{s}{a}(b-c)^2\rightarrow(1)$$

$$\text{Now, } n_a\overset{?}{\ge}\frac{b^2-bc+c^2}{2R}\Leftrightarrow\frac{n_a}{h_a}\overset{?}{\ge}\frac{b^2-bc+c^2}{bc}\Leftrightarrow\frac{n_a^2}{h_a^2}-1\overset{?}{\ge}\left(\frac{b^2-bc+c^2}{bc}\right)^2-1$$
$$=\frac{(b-c)^2\left(b^2+c^2\right)}{b^2c^2}\Leftrightarrow\frac{n_a^2-h_a^2}{h_a^2}\overset{?}{\ge}\frac{(b-c)^2\left(b^2+c^2\right)}{b^2c^2}\overset{\text{via }(1)}{\Leftrightarrow}$$
$$s(s-a)+\frac{s}{a}(b-c)^2-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}\overset{?}{\ge}\frac{(b-c)^2\left(b^2+c^2\right)}{b^2c^2}\cdot\frac{b^2c^2}{4R^2}$$
$$\Leftrightarrow\left(\frac{s}{a}+\frac{s(s-a)}{a^2}\right)(b-c)^2\overset{?}{\ge}\frac{(b-c)^2\left(b^2+c^2\right)}{4R^2}\Leftrightarrow\frac{s^2}{a^2}\overset{?}{\ge}\frac{b^2+c^2}{4R^2}\ (\because(b-c)^2\ge 0)$$
$$\Leftrightarrow 4R^2s^2\overset{?}{\ge}a^2b^2+c^2a^2\rightarrow\text{true (strict inequality)}\because 4R^2s^2\overset{\text{Goldstone}}{\ge}$$
$$\sum_{\text{cyc}}a^2b^2>a^2b^2+c^2a^2\ \therefore\ n_a\ge\frac{b^2-bc+c^2}{2R}\Rightarrow\frac{m_an_a}{h_a^2}\overset{\text{Tereshin}}{\ge}\frac{\frac{b^2+c^2}{4R}\cdot\frac{b^2-bc+c^2}{2R}}{\frac{b^2c^2}{4R^2}}$$
$$=\frac{\left(b^2+c^2\right)\left(b^2-bc+c^2\right)}{2b^2c^2}\Rightarrow\frac{m_an_a}{h_a^2}-1-\frac{(b-c)^4}{2b^2c^2}\ge$$
$$\frac{\left(b^2+c^2\right)\left(b^2-bc+c^2\right)-2b^2c^2-(b-c)^4}{2b^2c^2}=\frac{3bc(b-c)^2}{2b^2c^2}=\frac{3(b-c)^2}{2bc}$$
$$>\frac{\sqrt{2}(b-c)^2}{bc}\Rightarrow\frac{m_an_a}{h_a^2}\ge 1+\frac{(b-c)^4}{2b^2c^2}+\frac{\sqrt{2}(b-c)^2}{bc}=\left(1+\frac{(b-c)^2}{\sqrt{2}bc}\right)^2$$
$$\Rightarrow\frac{\sqrt{m_an_a}}{h_a}\ge 1+\frac{(b-c)^2}{\sqrt{2}bc}\text{ and analogs}\ \therefore\ \frac{\sqrt{m_an_a}}{h_a}+\frac{\sqrt{m_bn_b}}{h_b}+\frac{\sqrt{m_cn_c}}{h_c}\ge$$
$$3+\frac{1}{\sqrt{2}}\sum_{\text{cyc}}\frac{b^2+c^2-2bc}{bc}=3+\frac{1}{\sqrt{2}}\left(\sum_{\text{cyc}}\frac{b+c}{a}-6\right)$$
$$\therefore\ \frac{\sqrt{m_an_a}}{h_a}+\frac{\sqrt{m_bn_b}}{h_b}+\frac{\sqrt{m_cn_c}}{h_c}\ge\frac{\sqrt{2}}{2}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-3\sqrt{2}$$

$\forall\ \Delta$ ABC, with equality iff Δ ABC is equilateral (QED)