In any Δ ABC with p_a, p_b, p_c

$\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{\sqrt{m_a p_a}}{h_a}+\frac{\sqrt{m_b p_b}}{h_b}+\frac{\sqrt{m_c p_c}}{h_c} \geq \frac{\sqrt{3}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-2\sqrt{3}$$

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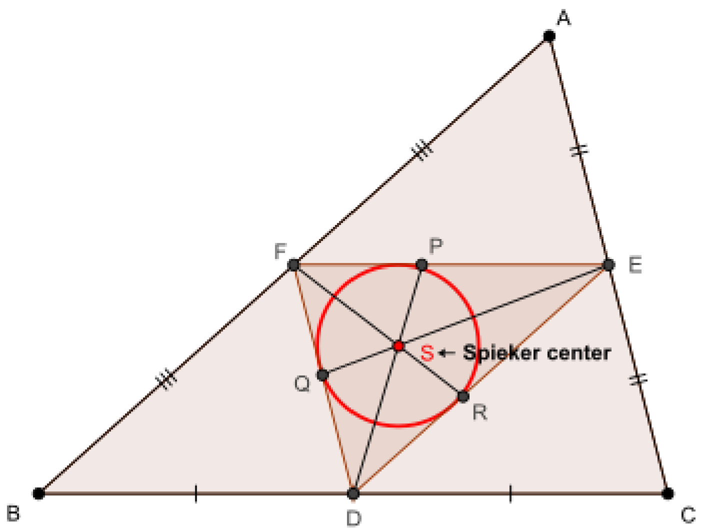

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'(say)$

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4} \Rightarrow r'=\frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $arrive$ at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$\text{(i)},(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{\text{(ii)}}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$
$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$
$$\therefore\ \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$
$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$
$$\text{Now, } p_a\ge\frac{2b^2-bc+2c^2}{6R}\Leftrightarrow\frac{p_a^2}{h_a^2}\ge\frac{(2b^2-bc+2c^2)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$
$$\Leftrightarrow\frac{p_a^2-h_a^2}{h_a^2}\ge\left(\frac{2b^2-bc+2c^2}{3bc}-1\right)\left(\frac{2b^2-bc+2c^2}{3bc}+1\right)$$
$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}\frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}}{h_a^2}\ge\frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2\left(b^2+bc+c^2\right)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \quad \left(\because (b-c)^2 \geq 0\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \left(b^2+bc+c^2\right)(s-b)(s-c)$$

$$= \left(b^2+bc+c^2\right)\left(-s(s-a)+bc\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \left(bc-s(s-a)\right)\left(b^2+c^2\right)+b^2c^2-bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \geq \left(bc-s(s-a)\right)\left((2s-a)^2-2bc\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \geq$$
$$\left(bc-s(s-a)\right)(2s-a)^2-2b^2c^2+2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}+b^2c^2+s(s-a)(2s-a)^2-bc\left(s(s-a)+(2s-a)^2\right) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3-12sa^2-4a^3}{4(s-a)(2s+a)^2}.b^2c^2-\left(5s^2-5sa+a^2\right).bc+s(s-a)(2s-a)^2 \overset{(*)}{\geq} 0}$$

$$\text{Now, LHS of } (*) \text{ is a quadratic polynomial with discriminant} =$$

$$\left(5s^2-5sa+a^2\right)^2-\frac{25s^3-12sa^2-4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2\left(12s^4-18s^3a+5s^2a^2+2sa^3-a^4\right)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)\left(12s^2+6sa+5a^2\right)+6a^3\right)}{(2s+a)^2} < 0 \ (\because s > a)$$

$$\therefore (*) \text{ is true (strict inequality)}$$

$$\therefore p_a \geq \frac{2b^2-bc+2c^2}{6R} \Rightarrow \frac{m_ap_a}{h_a^2} \overset{\text{Tereshin}}{\geq} \frac{\frac{b^2+c^2}{4R}.\frac{2b^2-bc+2c^2}{6R}}{\frac{b^2c^2}{4R^2}}$$

$$\Rightarrow \frac{m_ap_a}{h_a^2}-1-\frac{(b-c)^4}{3b^2c^2} \geq \frac{\left(2b^2-bc+2c^2\right)\left(b^2+c^2\right)-6b^2c^2-2(b-c)^4}{6b^2c^2}$$

$$= \frac{7bc(b-c)^2}{6b^2c^2} \geq \frac{2(b-c)^2}{\sqrt{3}bc} \therefore \frac{m_ap_a}{h_a^2} \geq 1+\frac{(b-c)^4}{3b^2c^2}+\frac{2(b-c)^2}{\sqrt{3}bc} = \left(1+\frac{(b-c)^2}{\sqrt{3}bc}\right)^2$$

$$\Rightarrow \frac{\sqrt{m_ap_a}}{h_a} \geq 1+\frac{(b-c)^2}{\sqrt{3}bc} \text{ and analogs } \therefore \frac{\sqrt{m_ap_a}}{h_a}+\frac{\sqrt{m_bp_b}}{h_b}+\frac{\sqrt{m_cp_c}}{h_c} \geq$$

$$3+\frac{1}{\sqrt{3}}.\sum_{cyc}\frac{b^2+c^2-2bc}{bc} = 3+\frac{1}{\sqrt{3}}.\left(\sum_{cyc}\frac{b+c}{a}-6\right)$$

$$\therefore \frac{\sqrt{m_ap_a}}{h_a}+\frac{\sqrt{m_bp_b}}{h_b}+\frac{\sqrt{m_cp_c}}{h_c} \geq \frac{\sqrt{3}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right)+3-2\sqrt{3}$$

$$\forall \Delta ABC, \text{ with equality iff } \Delta ABC \text{ is equilateral (QED)}$$