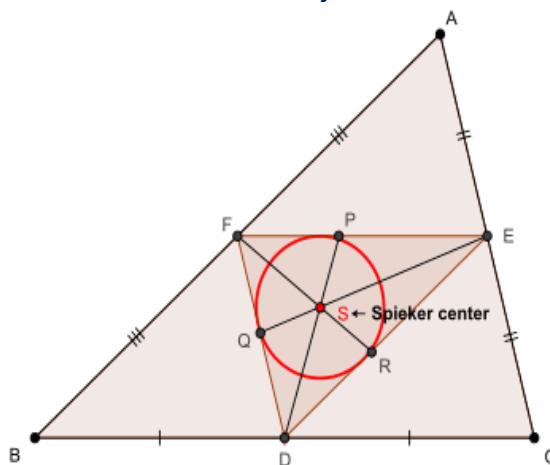**In** *any* Δ **ABC with** p_a, p_b, p_c
$\rightarrow$ **Spieker cevians, the** fo*llowing* re*lationship holds* :

$$\frac{\sqrt{p_a n_a}}{h_a} + \frac{\sqrt{p_b n_b}}{h_b} + \frac{\sqrt{p_c n_c}}{h_c} \geq \frac{\sqrt{6}}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 3 - 2\sqrt{6}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

Let AS produced meet BC *at* **X** *and* **m($\sphericalangle$BAX)** $= \alpha$ *and* **m($\sphericalangle$CAX)** $= \beta$ (*say*)
and **in***radius of* Δ **DEF** $= r'(say)$

$$\text{Now,} \; 16[\text{DEF}]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2 b^2 - \sum a^4\right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [\text{DEF}] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$\because$ **Spieker center is incenter of** Δ **DEF,** $\therefore$ **m($\sphericalangle$AFS)** $= B + \dfrac{C}{2} = \dfrac{2B+C}{2} = \dfrac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \; and \; \text{m($\sphericalangle$AES)} = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via **(1), (2)** *and* **using cosine law on** Δ **AFS** *and* Δ **AES, we** *arrive* at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$- \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,} \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a \cdot 2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$\text{(i), (*), (**)} \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3 - abc + a(2b^2 + 2c^2 - a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta \text{ AFS,} \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta \text{ AES, } b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via (***) and (****)}}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\cdot)}{=} \frac{2s}{(2s+a)^2}\left(b^3 + c^3 - abc + a(4m_a^2)\right)}$$

$$\text{Now, } b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)$$

$$= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2)$$

$$= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2)$$

$$= (2s+a)(b^2 - bc + c^2) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right)$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\cdot\frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\cdot$$

$$\frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$

$$- \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a)\cdot\frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2}\right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3 + c^3 - abc + a(4m_a^2) \overset{(\cdot\cdot)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2}\right) + \frac{s(b-c)^2}{2}}$$

$$\therefore (\cdot), (\cdot\cdot) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s-a) + (b-c)^2\left(\left(\frac{s}{2s+a}\right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2\cdot\left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\cdot\cdot\cdot)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{\left(2b^2 - bc + 2c^2\right)^2}{36R^2\cdot\frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{via\ (\cdots)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2 + bc + c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2 h_a^2 (2s+a)^2} \geq \frac{4(b-c)^2(b^2 + bc + c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2 + bc + c^2}{9b^2c^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2 + bc + c^2)(s-b)(s-c)$$
$$= (b^2 + bc + c^2)(-s(s-a) + bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (bc - s(s-a))(b^2 + c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq (bc - s(s-a))((2s-a)^2 - 2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$
$$(bc - s(s-a))(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc(s(s-a) + (2s-a)^2) \geq 0$$

$$\Leftrightarrow \boxed{\boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2} \cdot b^2c^2 - (5s^2 - 5sa + a^2) \cdot bc + s(s-a)(2s-a)^2 \overset{(\square)}{\geq} 0}}$$

Now, LHS of $(\square)$ is a quadratic polynomial with discriminant $=$

$$(5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2} \cdot s(2s-a)^2$$

$$= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)(12s^2 + 6sa + 5a^2) + 6a^3\right)}{(2s+a)^2} < 0 \ (\because s > a)$$

$$\therefore (\square) \text{ is true (strict inequality)}$$

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \text{ and analogs} \to (m)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2 + c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - s(a^2 - (b-c)^2) = as(s-a) + s(b-c)^2$$

$$\Rightarrow n_a^2 = s(s-a) + \frac{s}{a}(b-c)^2 \to (l)$$

$$\text{Now, } n_a \overset{?}{\geq} \frac{b^2 - bc + c^2}{2R} \Leftrightarrow \frac{n_a}{h_a} \overset{?}{\geq} \frac{b^2 - bc + c^2}{bc} \Leftrightarrow \frac{n_a^2}{h_a^2} - 1 \overset{?}{\geq} \left(\frac{b^2 - bc + c^2}{bc}\right)^2 - 1$$

$$= \frac{(b-c)^2(b^2+c^2)}{b^2 c^2} \Leftrightarrow \frac{n_a^2 - h_a^2}{h_a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2 c^2} \overset{via\ (l)}{\Leftrightarrow}$$

$$s(s-a) + \frac{s}{a}(b-c)^2 - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2} \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{b^2 c^2} \cdot \frac{b^2 c^2}{4R^2}$$

$$\Leftrightarrow \left(\frac{s}{a} + \frac{s(s-a)}{a^2}\right)(b-c)^2 \overset{?}{\geq} \frac{(b-c)^2(b^2+c^2)}{4R^2} \Leftrightarrow \frac{s^2}{a^2} \overset{?}{\geq} \frac{b^2+c^2}{4R^2} \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow 4R^2 s^2 \overset{?}{\geq} a^2 b^2 + c^2 a^2 \to \text{true (strict inequality) } \because 4R^2 s^2 \overset{\text{Goldstone}}{\geq}$$

$$\sum_{cyc} a^2 b^2 > a^2 b^2 + c^2 a^2 \therefore n_a \geq \frac{b^2 - bc + c^2}{2R} \text{ and analogs} \to (n)$$

$$\therefore via\ (m)\ and\ (n), \frac{p_a n_a}{h_a^2} \geq \frac{\frac{2b^2 - bc + 2c^2}{6R} \cdot \frac{b^2 - bc + c^2}{2R}}{\frac{b^2 c^2}{4R^2}} \Rightarrow \frac{p_a n_a}{h_a^2} - 1 - \frac{2}{3} \cdot \frac{(b-c)^4}{b^2 c^2}$$

$$\geq \frac{(2b^2 - bc + 2c^2)(b^2 - bc + c^2) - 3b^2 c^2 - 2(b-c)^4}{3b^2 c^2} = \frac{5bc(b-c)^2}{3b^2 c^2}$$

$$\geq \frac{2\sqrt{6}(b-c)^2}{3bc} \therefore \frac{p_a n_a}{h_a^2} \geq 1 + \frac{2}{3} \cdot \frac{(b-c)^4}{b^2 c^2} + \frac{2\sqrt{6}(b-c)^2}{3bc} = \left(1 + \frac{\sqrt{6}(b-c)^2}{3bc}\right)^2$$

$$\Rightarrow \frac{\sqrt{p_a n_a}}{h_a} \geq 1 + \frac{\sqrt{6}(b-c)^2}{3bc} \text{ and analogs} \therefore \frac{\sqrt{p_a n_a}}{h_a} + \frac{\sqrt{p_b n_b}}{h_b} + \frac{\sqrt{p_c n_c}}{h_c} \geq$$

$$3 + \frac{\sqrt{6}}{3} \cdot \sum_{cyc} \frac{b^2 + c^2 - 2bc}{bc} = 3 + \frac{\sqrt{6}}{3} \cdot \left(\sum_{cyc} \frac{b+c}{a} - 6\right) \therefore \frac{\sqrt{p_a n_a}}{h_a} + \frac{\sqrt{p_b n_b}}{h_b} + \frac{\sqrt{p_c n_c}}{h_c}$$

$$\geq \frac{\sqrt{6}}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}\right) + 3 - 2\sqrt{6} \ \forall \ \Delta\ ABC,$$

with equality iff Δ ABC is equilateral (QED)