

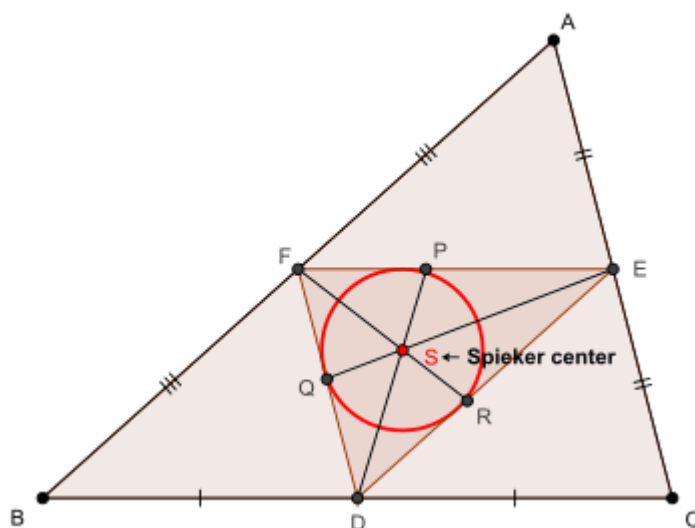
# ROMANIAN MATHEMATICAL MAGAZINE

In any  $\triangle ABC$  with  $p_a, p_b, p_c \rightarrow$  Spieker cevians, the following relationship holds :

$$\frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \geq \frac{2}{3} \left( \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \right) - 1$$

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Let AS produced meet BC at X and  $m(\angle BAX) = \alpha$  and  $m(\angle CAX) = \beta$  (say)  
and inradius of  $\triangle DEF = r'$  (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left( \frac{a^2}{4} \right) \left( \frac{b^2}{4} \right) - \sum \frac{a^4}{16} = \frac{1}{16} \left( 2 \sum a^2 b^2 - \sum a^4 \right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left( \frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2} \right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\begin{aligned} \because \text{Spieker center is incenter of } \triangle DEF, \therefore m(\angle AFS) &= B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2} \\ &= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\angle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on  $\triangle AFS$  and  $\triangle AES$ , we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4\sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left( \frac{2r}{2\sin \frac{C}{2}} \right) \left( \frac{c}{2} \right) \sin \frac{A-B}{2} \\ &= \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left( \frac{2r}{2\sin \frac{B}{2}} \right) \left( \frac{b}{2} \right) \sin \frac{A-C}{2} \\ \Rightarrow 2AS^2 &\stackrel{(i)}{=} \frac{r^2}{4\sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left( \frac{2r}{2\sin \frac{C}{2}} \right) \left( \frac{c}{2} \right) \sin \frac{A-B}{2} + \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{b^2}{4} \end{aligned}$$

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$$\begin{aligned}
 & - \left( \frac{2r}{2\sin\frac{B}{2}} \right) \left( \frac{b}{2} \right) \sin\frac{A-C}{2} \\
 \text{Now, } & \left( \frac{2r}{2\sin\frac{C}{2}} \right) \left( \frac{c}{2} \right) \sin\frac{A-B}{2} + \left( \frac{2r}{2\sin\frac{B}{2}} \right) \left( \frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & = \frac{r}{2} \left( 4R\cos\frac{C}{2} \sin\frac{A-B}{2} + 4R\cos\frac{B}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left( 2\sin\frac{A+B}{2} \sin\frac{A-B}{2} + 2\sin\frac{A+C}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left( 1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2 \left( 1 - 2\sin^2\frac{A}{2} \right) \right) \\
 & = 2Rr \left( \frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 & = \frac{Rr}{8Rrs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 & = \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a \cdot 2bcc\cos A}{8s} = \frac{bc \left( (2s-a)\sin^2\frac{A}{2} - a \left( 1 - 2\sin^2\frac{A}{2} \right) \right)}{2s} \\
 & = \frac{bc \left( (2s+a)\sin^2\frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 & \Rightarrow - \left( \frac{2r}{2\sin\frac{C}{2}} \right) \left( \frac{c}{2} \right) \sin\frac{A-B}{2} - \left( \frac{2r}{2\sin\frac{B}{2}} \right) \left( \frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \\
 \text{Again, } & \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4} \left( \frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 & = \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} \\
 & \text{(i), (*), (**)} \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 & = \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{4} \\
 & = \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s} \\
 \text{Via sine law on } \triangle AFS, & \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{4s}{(a+b)\sin\frac{C}{2}} \\
 \Rightarrow c\sin\alpha & \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b\sin\beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS} \\
 \text{Now, [BAX] + [BAX]} & = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs \\
 \text{via (***) and (***)} & \Rightarrow \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS
 \end{aligned}$$

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$$\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s}$$

$$\therefore p_a^2 \stackrel{(\bullet)}{=} \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2))$$

Now,  $b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)$

$$= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2)$$

$$= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2)$$

$$= (2s+a)(b^2 - bc + c^2) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right)$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)((z+x) + (x+y) - 2(y+z))}{4}$$

$$- \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left( s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left( s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a) \left( s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore b^3 + c^3 - abc + a(4m_a^2) \stackrel{(\bullet\bullet)}{=} (2s+a) \left( \frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2} \left( \frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2} \right)$$

$$= s(s-a) + (b-c)^2 \left( \left( \frac{s}{2s+a} \right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4} \right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left( \frac{s}{2s+a} + \frac{1}{2} \right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4} \left( \frac{(4s+a)^2}{(2s+a)^2} - 1 \right)$$

$$\Rightarrow p_a^2 \stackrel{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\begin{aligned}
 & \text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2 - bc + 2c^2)^2}{36R^2 \cdot \frac{b^2c^2}{4R^2}} \\
 & \Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left( \frac{2b^2 - bc + 2c^2}{3bc} - 1 \right) \left( \frac{2b^2 - bc + 2c^2}{3bc} + 1 \right) \\
 & \text{via (...)} \Leftrightarrow \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2 + bc + c^2)}{9b^2c^2} \\
 & \Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2 + bc + c^2)}{9b^2c^2} \\
 & \Leftrightarrow \frac{4s(s-a)(s-b)(s-c)(2s+a)^2}{9s^3b^2c^2} \geq \frac{b^2 + bc + c^2}{9b^2c^2} \quad (\because (b-c)^2 \geq 0) \\
 & \Leftrightarrow \frac{4(s-a)(2s+a)^2}{4(s-a)(2s+a)^2} \geq (b^2 + bc + c^2)(s-b)(s-c) \\
 & \quad = (b^2 + bc + c^2)(-s(s-a) + bc) \\
 & \Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (bc - s(s-a))(b^2 + c^2) + b^2c^2 - bcs(s-a) \\
 & \Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq (bc - s(s-a))((2s-a)^2 - 2bc) \\
 & \Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq \\
 & \quad (bc - s(s-a))(2s-a)^2 - 2b^2c^2 + 2bcs(s-a) \\
 & \Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc(s(s-a) + (2s-a)^2) \geq 0 \\
 & \Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2} \cdot b^2c^2 - (5s^2 - 5sa + a^2) \cdot bc + s(s-a)(2s-a)^2 \geq 0} \quad (\text{Q})
 \end{aligned}$$

Now, LHS of (Q) is a quadratic polynomial with discriminant =

$$\begin{aligned}
 & (5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2} \cdot s(2s-a)^2 \\
 & = \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2} \\
 & = \frac{-a^2(s-a)((s-a)(12s^2 + 6sa + 5a^2) + 6a^3)}{(2s+a)^2} < 0 \quad (\because s > a)
 \end{aligned}$$

$\therefore$  (Q) is true (strict inequality)

$$\begin{aligned}
 \therefore p_a & \geq \frac{2b^2 - bc + 2c^2}{6R} \Rightarrow \frac{p_a}{h_a} + \frac{1}{3} \geq \frac{2b^2 - bc + 2c^2}{3bc} + \frac{1}{3} = \frac{2b^2 + 2c^2}{3bc} \\
 & \Rightarrow \frac{p_a}{h_a} + \frac{1}{3} \geq \frac{2}{3} \left( \frac{b}{c} + \frac{c}{b} \right) \text{ and analogs } \therefore \frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} + 1 \geq \frac{2}{3} \sum_{\text{cyc}} \left( \frac{b}{c} + \frac{c}{b} \right) \\
 & = \frac{2}{3} \left( \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \right) \Rightarrow \frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \geq \frac{2}{3} \left( \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \right) - 1 \\
 & \quad \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}
 \end{aligned}$$