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In any Δ ABC with p_a, p_b, p_c
$\to$ Spieker cevians, the following relationship holds :

$$\frac{\sqrt{p_b p_c}}{h_a}+\frac{\sqrt{p_c p_a}}{h_b}+\frac{\sqrt{p_a p_b}}{h_c}\ge \frac{\sqrt{6}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3$$

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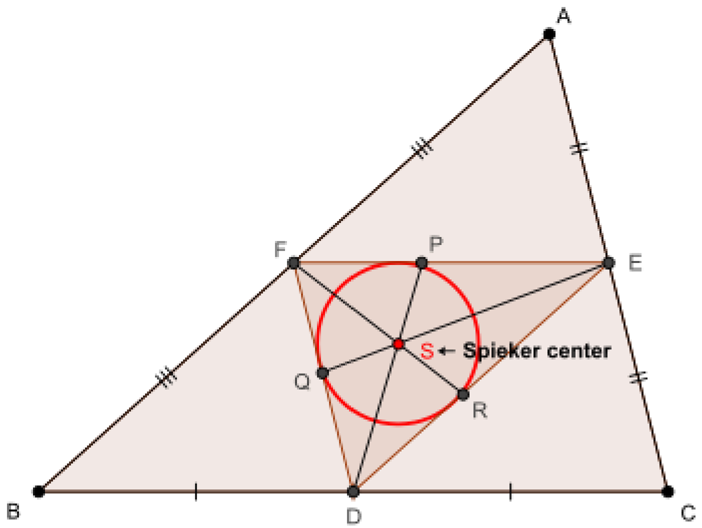

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\text{ DEF}, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS,}\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2 - bc + 2c^2)^2}{36R^2 . \frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (bc - s(s-a))(b^2+c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq (bc - s(s-a))((2s-a)^2 - 2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$

$$(bc - s(s-a))(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc(s(s-a) + (2s-a)^2) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2}.b^2c^2 - (5s^2 - 5sa + a^2).bc + s(s-a)(2s-a)^2 \overset{(\blacksquare)}{\geq} 0}$$

Now, LHS of $(\blacksquare)$ is a quadratic polynomial with discriminant $=$

$$(5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)(12s^2 + 6sa + 5a^2) + 6a^3\right)}{(2s+a)^2} < 0 \ (\because s > a)$$

$\therefore (\blacksquare)$ is true (strict inequality)

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \text{ and analogs} \rightarrow (m)$$

$$\therefore p_b p_c \geq \frac{(2c^2 - ca + 2a^2)(2a^2 - ab + 2b^2)}{36R^2} \overset{?}{\geq} \frac{(4a^2 + 4bc - ab - ac)^2}{144R^2}$$

$$\Leftrightarrow 15a^2(b-c)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \therefore p_b p_c \geq \frac{(4a^2 + 4bc - ab - ac)^2}{144R^2}$$

$$\Rightarrow \sqrt{p_b p_c} \geq \frac{4a^2 + 4bc - ab - ac}{12R} \text{ and analogs} \rightarrow (n)$$

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$$\text{Now, } \frac{\sqrt{6}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3=\frac{\sqrt{6}}{3}\left(\frac{s^2-2Rr+r^2}{2Rr}-6\right)+3$$

$$=\frac{\sqrt{6}}{3}\left(\frac{s^2-14Rr+r^2}{2Rr}\right)+3 \therefore \left(\frac{\sqrt{6}}{3}\left(\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}-6\right)+3\right)^2$$

$$=\frac{(s^2-14Rr+r^2)^2}{6R^2r^2}+9+\frac{\sqrt{6}}{Rr}(s^2-14Rr+r^2)$$

$$=\frac{(s^2-14Rr+r^2)^2+54R^2r^2+6\sqrt{6}Rr(s^2-14Rr+r^2)}{6R^2r^2}$$

$$\le\frac{(s^2-14Rr+r^2)^2+54R^2r^2+15Rr(s^2-14Rr+r^2)}{6R^2r^2}$$

$$\left(\because s^2-14Rr+r^2 \overset{\text{Gerretsen + Euler}}{\ge} 0 \text{ and } 225>216 \Rightarrow 15>6\sqrt{6}\right)$$

$\therefore$ in order to prove the original inequality, it suffices to prove :

$$\boxed{\sum_{cyc}\frac{p_bp_c}{h_a^2}+2\sum_{cyc}\left(\sqrt{\frac{p_bp_c}{h_bh_c}}.p_a\right) \overset{(\blacksquare\blacksquare)}{\ge} \frac{(s^2-14Rr+r^2)^2+54R^2r^2+15Rr(s^2-14Rr+r^2)}{6R^2r^2}}$$

$$\text{Now, via (m) and (n), } \sum_{cyc}\frac{p_bp_c}{h_a^2}+2\sum_{cyc}\left(\sqrt{\frac{p_bp_c}{h_bh_c}}.p_a\right)\ge$$

$$\sum_{cyc}\frac{a^2(2c^2-ca+2a^2)(2a^2-ab+2b^2)}{36R^2.4r^2s^2}+$$

$$2\sum_{cyc}\left(\frac{4a^2+4bc-ab-ac}{12R}.\frac{2b^2-bc+2c^2}{6R}.\frac{bc}{4r^2s^2}\right)$$

$$=\frac{4\sum_{cyc}a^6-2\sum_{cyc}\left(ab\left(\sum_{cyc}a^4-c^4\right)\right)+12\sum_{cyc}\left(a^2b^2\left(\sum_{cyc}a^2-c^2\right)\right)-3abc\sum_{cyc}a^3}{144R^2r^2s^2}$$

$$+\frac{-4\sum_{cyc}a^3b^3+5abc\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)-3abc\right)}{144R^2r^2s^2}$$

$$=\frac{8\left(3s^6-(36Rr+11r^2)s^4+r^2s^2(148R^2+116Rr+13r^2)-5r^3(4R+r)^3\right)}{144R^2r^2s^2}$$

$$\left(\begin{array}{c}\text{using } \sum_{cyc}a^6=4s^2(s^2-6Rr-3r^2)^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right),\\ \sum_{cyc}a^4=2\sum_{cyc}a^2b^2-16r^2s^2, \sum_{cyc}a^2b^2=(s^2+4Rr+r^2)^2-16Rrs^2 \text{ and}\\ \sum_{cyc}a^3b^3=(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\end{array}\right)$$

$$\overset{?}{\ge}\frac{(s^2-14Rr+r^2)^2+54R^2r^2+15Rr(s^2-14Rr+r^2)}{6R^2r^2}$$

$$\Leftrightarrow (3R-17r)s^4+rs^2(28R^2+155Rr+10r^2)-5r^2(4R+r)^3 \underset{(\blacksquare\blacksquare\blacksquare)}{\overset{?}{\ge}} 0$$

$$\text{Now, } (3R-17r)s^4+rs^2(28R^2+155Rr+10r^2)$$

$$= (3R-6r)s^4 - 11rs^4 + rs^2(28R^2+155Rr+10r^2) \overset{\text{Gerretsen}}{\geq}$$
$$(3R-6r)(16Rr-5r^2)s^2 - 11r(4R^2+4Rr+3r^2)s^2 + rs^2(28R^2+155Rr+10r^2)$$
$$= r(32R^2+7r^2)s^2 \overset{?}{\geq} 5r^2(4R+r)^3 \Leftrightarrow 48t^3-100t^2+13t-10 \overset{?}{\geq} 0 \ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)(46t^2+2t(t-2)+5) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\blacksquare\blacksquare\blacksquare) \Rightarrow (\blacksquare\blacksquare) \text{ is true}$$
$$\therefore \frac{\sqrt{p_b p_c}}{h_a} + \frac{\sqrt{p_c p_a}}{h_b} + \frac{\sqrt{p_a p_b}}{h_c} \geq \frac{\sqrt{6}}{3}\left(\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} - 6\right) + 3$$
$\forall \Delta ABC$, with equality iff ΔABC is equilateral (QED)