ROMANIAN MATHEMATICAL MAGAZINE

If $n_a, n_b, n_c \to$ Nagel cevians, $g_a, g_b, g_c \to$ Gergonne's cevians in Δ ABC, then

$$n_a g_a + n_b g_b + n_c g_c \ge s^2 . \sqrt{1 + \frac{16r^2(R-2r)}{s^2 R}}$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \text{ and}$$
$$b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c) \text{ \& adding the two, we get :}$$
$$(b^2+c^2)(2s-b-c) = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$
$$= 2a(n_a^2+g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2)$$
$$= 2(n_a^2+g_a^2) + a^2 - (b-c)^2 \Rightarrow 2(b^2+c^2) - a^2 + (b-c)^2 = 2(n_a^2+g_a^2)$$
$$\Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2+g_a^2) \Rightarrow 2(b-c)^2 + 4s(s-a) = 2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2 + g_a^2 \overset{(*)}{=} (b-c)^2 + 2s(s-a)$$
$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$
$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$
$$= an_a^2 + a(as-s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$
$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 \overset{(**)}{=} s\left(s-a+\frac{(b-c)^2}{a}\right)$$
$$\text{Via } (*) \text{ and } (**), g_a^2 = (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a}$$
$$= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a}$$
$$= (s-a)^2 + (b-c+a)(b-c-a) + \frac{4s(s-b)(s-c)}{a}$$
$$= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a}$$
$$= (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a}-1\right) = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a}$$
$$= (s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right) \Rightarrow g_a^2 \overset{(***)}{=} (s-a)\left(s-\frac{(b-c)^2}{a}\right)$$
$$\therefore (**), (***) \Rightarrow n_a^2 g_a^2 = s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$
$$= s(s-a)\left(s(s-a) + s.\frac{(b-c)^2}{a} - \frac{(b-c)^2}{a}(s-a) - \frac{(b-c)^4}{a^2}\right)$$
$$= s(s-a)\left(s(s-a) + (b-c)^2 - \frac{(b-c)^4}{a^2}\right)$$
$$= s^2(s-a)^2 + (b-c)^2.\frac{4s(s-a)(s-b)(s-c)}{a^2} = s^2(s-a)^2 + (b-c)^2.\frac{4F^2}{a^2}$$

$$\Rightarrow n_a^2 g_a^2 = s^2(s-a)^2 + (b-c)^2.h_a^2 \text{ and analogs} \to (1) \therefore n_a^2 g_a^2.n_b^2 g_b^2 =$$
$$(s^2(s-a)^2 + (b-c)^2.h_a^2).(s^2(s-b)^2 + (c-a)^2.h_b^2)$$
$$\geq \left(s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b\right)^2$$
$$\therefore \boxed{n_a g_a.n_b g_b \geq s^2(s-a)(s-b) + |b-c||c-a|.h_a h_b} \text{ and analogs} \to (2)$$
$$\therefore \sum_{cyc}(n_a g_a.n_b g_b) \geq s^2 \sum_{cyc}(s-a)(s-b) + \sum_{cyc}(|b-c||c-a|.h_a h_b)$$
$$= s^2(4Rr + r^2) + \frac{4Rrs}{4R^2}\sum_{cyc}(c.|b-c||c-a|) \overset{\text{Triangle inequality}}{\geq}$$
$$s^2(4Rr + r^2) + \frac{rs}{R}.\left|\sum_{cyc}(c(b-c)(c-a))\right|$$
$$= s^2(4Rr + r^2) + \frac{rs}{R}.\left|\sum_{cyc} bc^2 - 3abc - \sum_{cyc} c^3 + \sum_{cyc} c^2 a\right|$$
$$= s^2(4Rr + r^2) + \frac{rs}{R}.\left|2s(s^2 + 4Rr + r^2) - 24Rrs - 2s(s^2 - 6Rr - 3r^2)\right|$$
$$= s^2(4Rr + r^2) + \frac{2rs^2}{R}.\left|-2Rr + 4r^2\right| = s^2(4Rr + r^2) + \frac{2rs^2}{R}.(2Rr - 4r^2)$$
$$\left(\because 2Rr \overset{\text{Euler}}{\geq} 4r^2\right) \therefore \left(\sum_{cyc} n_a g_a\right)^2 \geq \sum_{cyc} n_a^2 g_a^2 + 2s^2(4Rr + r^2) +$$
$$\frac{4rs^2}{R}.(2Rr - 4r^2) \overset{\text{via (1) and analogs}}{=} s^2\sum_{cyc}(s-a)^2 + 4r^2 s^2 \sum_{cyc}\frac{(b-c)^2}{a^2}$$
$$+2s^2(4Rr + r^2) + \frac{4rs^2}{R}.(2Rr - 4r^2)$$
$$= s^2\left(3s^2 - 4s^2 + 2(s^2 - 4Rr - r^2)\right) + \frac{4r^2 s^2}{16R^2 r^2 s^2}.\sum_{cyc}\left(b^2c^2\left(\sum_{cyc} a^2 - a^2 - 2bc\right)\right)$$
$$+2s^2(4Rr + r^2) + \frac{4rs^2}{R}.(2Rr - 4r^2)$$
$$= s^2(s^2 - 8Rr - 2r^2) + \frac{1}{4R^2}.\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 48R^2r^2s^2 - 2\sum_{cyc} a^3b^3\right)$$
$$+2s^2(4Rr + r^2) + \frac{4rs^2}{R}.(2Rr - 4r^2) = s^2(s^2 - 8Rr - 2r^2) +$$
$$\frac{4r^2}{4R^2}.\left(-s^4 + (12R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3\right) + 2s^2(4Rr + r^2)$$
$$+\frac{4rs^2}{R}.(2Rr - 4r^2)\left(\begin{array}{c}\text{using } \sum_{cyc} a^2b^2 = (s^2 + 4Rr + r^2)^2 - 16Rrs^2 \text{ and} \\ \sum_{cyc} a^3b^3 = (s^2 + 4Rr + r^2)^3 - 24Rrs^2(s^2 + 2Rr + r^2)\end{array}\right)$$
$$= s^4 + \frac{r^2\left(-s^4 + (12R^2 + 4Rr - 2r^2)s^2 - r(4R + r)^3\right) + 4Rrs^2(2Rr - 4r^2)}{R^2} \overset{?}{\geq}$$

$$s^4\left(1+\frac{16r^2(R-2r)}{s^2R}\right)=s^4+\frac{16r^2s^2(R-2r)}{R}$$

$$\Leftrightarrow s^4-\left(4R^2+20Rr-2r^2\right)s^2+r(4R+r)^3\underset{(\bullet)}{\overset{?}{\leq}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0$$

$$\Rightarrow s^4-s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3\leq 0\Rightarrow(\bullet)\text{ is true}$$

$$\Rightarrow\left(\sum_{cyc}n_a g_a\right)^2\geq s^4\left(1+\frac{16r^2(R-2r)}{s^2R}\right)$$

$$\therefore n_a g_a+n_b g_b+n_c g_c\geq s^2.\sqrt{1+\frac{16r^2(R-2r)}{s^2R}}$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)