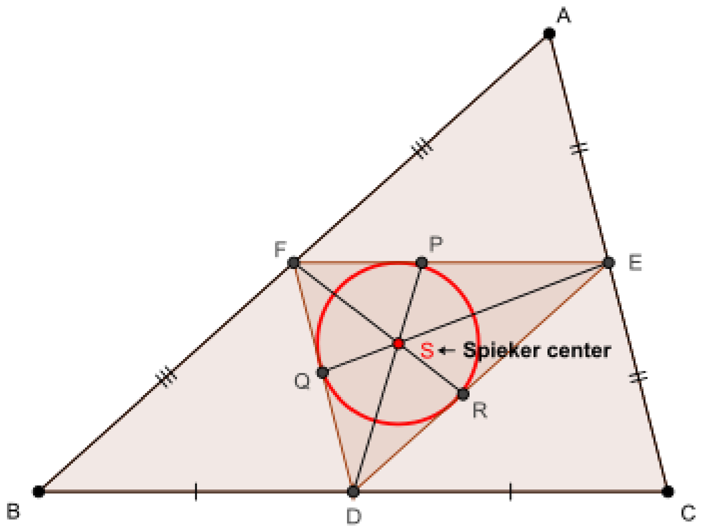
ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC with p_a, p_b, p_c
$\to$ Spieker cevians, the following relationship holds :

$$\frac{p_a p_b p_c}{r_a r_b r_c} \le \frac{8R-7r}{9r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of $\Delta DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$\because$ Spieker center is incenter of ΔDEF, $\therefore$ $m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on ΔAFS and ΔAES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\blacksquare)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{We have: } \prod_{cyc}(2s+a) = 8s^3 + 4s^2\sum_{cyc} a + 2s\sum_{cyc} ab + 4Rrs$$

$$= 8s^3 + 4s^2.2s + 2s(s^2+4Rr+r^2) + 4Rrs$$

$$\Rightarrow \prod_{cyc}(2s+a) \overset{(\blacksquare\blacksquare)}{=} 2s(9s^2+6Rr+r^2)$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$= \sum_{cyc} a^3 - abc + 2a.2bc\cos A = 2s(s^2-6Rr-3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2) = 2s(Q+8Rr\cos A) \text{ and analogs}$$

$$(Q = s^2-8Rr-3r^2) \Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a(4m_a^2)\right)$$

$$= 8s^3\left(Q^3 + Q^2.8Rr\sum_{cyc}\cos A + Q.64R^2r^2.\sum_{cyc}\cos B\cos C + 512R^3r^3\prod_{cyc}\cos A\right)$$

$$= 8s^3\left(\begin{array}{c} Q^3 + Q^2.8Rr.\frac{R+r}{R} + Q.32R^2r^2.\left(\left(\frac{R+r}{R}\right)^2 - \left(3 - \frac{2(s^2-4Rr-r^2)}{4R^2}\right)\right) \\ +512R^3r^3.\frac{s^2-(2R+r)^2}{4R^2} \end{array}\right)$$

$$\Rightarrow \prod_{cyc}\left(b^3+c^3-abc+a(4m_a^2)\right) \overset{(\blacksquare\blacksquare\blacksquare)}{=}$$

$$8s^3\left(\begin{array}{c} (s^2-8Rr-3r^2)^3 + (s^2-8Rr-3r^2)^2.8r(R+r) + \\ (s^2-8Rr-3r^2).16r^2(s^2-4R^2+r^2) + 128Rr^3(s^2-(2R+r)^2) \end{array}\right)$$

$$\therefore (\blacksquare), (\blacksquare\blacksquare), (\blacksquare\blacksquare\blacksquare)$$

$$\Rightarrow \boxed{\prod_{cyc}\frac{p_a^2}{r_a^2} = \frac{8s^3.8s^3\begin{pmatrix}(s^2-8Rr-3r^2)^3+(s^2-8Rr-3r^2)^2.8r(R+r)+\\(s^2-8Rr-3r^2).16r^2(s^2-4R^2+r^2)+128Rr^3(s^2-(2R+r)^2)\end{pmatrix}}{4s^2(9s^2+6Rr+r^2)^2.r^2s^4}}$$

$$\overset{?}{\leq}\left(\frac{8R-7r}{9r}\right)^2$$

$$\Leftrightarrow \boxed{\begin{gathered}1296s^6-(5184R^2+11664Rr-15471r^2)s^4-rs^2\begin{pmatrix}6912R^3-10944R^2r+\\44748Rr^2+69570r^3\end{pmatrix}\\-r^2(2304R^4-3264R^3r+484R^2r^2+21212Rr^3+3937r^4)\underset{(\bullet)}{\overset{?}{\leq}}0\end{gathered}}$$

$$\text{Now, Rouche} \Rightarrow s^2-(m-n)\geq 0 \text{ and } s^2-(m+n)\leq 0, \text{where}$$

$$m = 2R^2+10Rr-r^2 \text{ and } n = 2(R-2r).\sqrt{R^2-2Rr}$$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0 \Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3 \overset{(\blacklozenge)}{\leq} 0$$

$$\therefore 1296s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\leq 0 \Rightarrow \text{in order to prove } (\bullet),$$

$$\text{it suffices to prove : LHS of } (\bullet)$$

$$\leq 1296s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)$$

$$\Leftrightarrow \boxed{\begin{gathered}(14256R+12879r)s^4-rs^2(51264R^2+60300Rr+70866r^2)\\-r(2304R^4-3264R^3r+484R^2r^2+21212Rr^3+3937r^4)\underset{(\bullet\bullet)}{\overset{?}{\leq}}0\end{gathered}} \textit{ and}$$

$$\because (14256R+12879r)(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3) \overset{\text{via }(\blacklozenge)}{\leq} 0$$

$$\therefore \text{in order to prove } (\bullet), \text{it suffices to prove :}$$

$$\text{LHS of } (\bullet\bullet)\leq(14256R+12879r)(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)$$

$$\Leftrightarrow \boxed{\begin{gathered}(8208R^3-71343R^2r-42192Rr^2+24156r^3)s^2\\+r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4)\overset{(\bullet\bullet\bullet)}{\geq}0\end{gathered}}$$

$\boxed{\textit{Case 1}}$ $8208R^3-71343R^2r-42192Rr^2+24156r^3\geq 0$ *and then* : LHS of $(\bullet\bullet\bullet)$

$$\geq r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4)>0$$

$$\Rightarrow (\bullet\bullet\bullet) \text{ is true}$$

$\boxed{\textit{Case 2}}$ $8208R^3-71343R^2r-42192Rr^2+24156r^3<0$ *and then* : LHS of $(\bullet\bullet\bullet)$

$$= -\left(-(8208R^3-71343R^2r-42192Rr^2+24156r^3)\right)s^2$$

$$+r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4)$$

$$\overset{\text{Gerretsen}}{\geq} -\left(-(8208R^3-71343R^2r-42192Rr^2+24156r^3)\right)(4R^2+4Rr+3r^2)$$

$$+r(228672R^4+376320R^3r+197437R^2r^2+47504Rr^3+4204r^4)\overset{?}{\geq}0$$

$$\Leftrightarrow 8208t^5-5967t^4-13299t^3-22184t^2+4388t+19168\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)(8208t^3+26865t^2+61329t+115672)+221760\Big) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet) \text{ is true} \therefore \text{combining both } cases,$$

$$\forall \Delta\, ABC, (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \Rightarrow \forall \Delta\, ABC, \prod_{cyc} \frac{p_a^2}{r_a^2} \leq \left(\frac{8R-7r}{9r}\right)^2$$

$$\Rightarrow \frac{p_a p_b p_c}{r_a r_b r_c} \leq \frac{8R-7r}{9r}\ \forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$