ROMANIAN MATHEMATICAL MAGAZINE

If $n_a, n_b, n_c \to$ Nagel cevians, $g_a, g_b, g_c \to$ Gergonne's cevians in Δ ABC, then

$$\frac{n_a g_a}{h_a}+\frac{n_b g_b}{h_b}+\frac{n_c g_c}{h_c} \geq \sqrt{16R^2+24Rr-31r^2}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c) \text{ and}$$
$$b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c) \text{ \& adding the two, we get :}$$
$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c) \Rightarrow 2a(b^2+c^2)$$
$$=2a\left(n_a^2+g_a^2\right)+a(a+b-c)(c+a-b) \Rightarrow 2(b^2+c^2)$$
$$=2\left(n_a^2+g_a^2\right)+a^2-(b-c)^2 \Rightarrow 2\left(b^2+c^2\right)-a^2+(b-c)^2=2\left(n_a^2+g_a^2\right)$$
$$\Rightarrow 4m_a^2+(b-c)^2=2\left(n_a^2+g_a^2\right) \Rightarrow 2(b-c)^2+4s(s-a)=2\left(n_a^2+g_a^2\right)$$
$$\Rightarrow n_a^2+g_a^2 \overset{(*)}{=} (b-c)^2+2s(s-a)$$
$$\text{Again, Stewart's theorem} \Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$$
$$\Rightarrow s\left(b^2+c^2\right)-bc(2s-a)=an_a^2+a\left(s^2-s(2s-a)+bc\right) \Rightarrow s\left(b^2+c^2\right)-2sbc$$
$$=an_a^2+a\left(as-s^2\right) \Rightarrow s\left(b^2+c^2-a^2-2bc\right)=an_a^2-as^2 \Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 \overset{(**)}{=} s\left(s-a+\frac{(b-c)^2}{a}\right)$$
$$\text{Via } (*) \text{ and } (**), g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$$
$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$
$$=(s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right) \Rightarrow g_a^2 \overset{(***)}{=} (s-a)\left(s-\frac{(b-c)^2}{a}\right)$$
$$\therefore (**),(***) \Rightarrow n_a^2 g_a^2=s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$
$$=s(s-a)\left(s(s-a)+s.\frac{(b-c)^2}{a}-\frac{(b-c)^2}{a}(s-a)-\frac{(b-c)^4}{a^2}\right)$$
$$=s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}\right)$$

$$= s^2(s-a)^2+(b-c)^2.\frac{4s(s-a)(s-b)(s-c)}{a^2} = s^2(s-a)^2+(b-c)^2.\frac{4F^2}{a^2}$$

$$\Rightarrow n_a^2 g_a^2 = s^2(s-a)^2+(b-c)^2.h_a^2 \text{ and analogs} \to (1) \therefore n_a^2 g_a^2.n_b^2 g_b^2 = \left(s^2(s-a)^2+(b-c)^2.h_a^2\right).\left(s^2(s-b)^2+(c-a)^2.h_b^2\right)$$

$$\ge \left(s^2(s-a)(s-b)+|b-c||c-a|.h_a h_b\right)^2$$

$$\therefore \boxed{n_a g_a.n_b g_b \ge s^2(s-a)(s-b)+|b-c||c-a|.h_a h_b} \text{ and analogs} \to (2)$$

$$\therefore \sum_{cyc}\left(\frac{n_a g_a}{h_a}.\frac{n_b g_b}{h_b}\right) \ge \sum_{cyc}\frac{s^2 ab(s-a)(s-b)}{4r^2s^2}+\sum_{cyc}(|b-c||c-a|) \overset{\text{Triangle inequality}}{\ge}$$

$$\frac{1}{4r^2}.\sum_{cyc} bc(-s^2+sa+bc)+\left|\sum_{cyc}((b-c)(c-a))\right|$$

$$= \frac{1}{4r^2}.\left(-s^2(s^2+4Rr+r^2)+12Rrs^2+\left(s^2+4Rr+r^2\right)^2-16Rrs^2\right)+$$

$$\left|\sum_{cyc} bc-\sum_{cyc} ab-\sum_{cyc} a^2+\sum_{cyc} ca\right| = \frac{(4R+r)^2+s^2}{4}+\left|\sum_{cyc} a^2-\sum_{cyc} ab\right|$$

$$\therefore \left(\sum_{cyc}\frac{n_a g_a}{h_a}\right)^2 \ge \sum_{cyc}\frac{n_a^2 g_a^2}{h_a^2}+\frac{(4R+r)^2+s^2}{2}+2\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)$$

$$\overset{\text{via } (1)}{=} \sum_{cyc}\frac{s^2a^2(s-a)^2}{4r^2s^2}+\sum_{cyc}(b-c)^2+\frac{(4R+r)^2+s^2}{2}+2\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)$$

$$= \frac{1}{4r^2}.\left(s^2\sum_{cyc} a^2-2s\sum_{cyc} a^3+2\sum_{cyc} a^2b^2-16r^2s^2\right)+\frac{(4R+r)^2+s^2}{2}$$

$$+4\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)$$

$$= \frac{1}{4r^2}.\left(2s^2(s^2-4Rr-r^2)-4s^2(s^2-6Rr-3r^2)+2\left(s^2+4Rr+r^2\right)^2 -32Rrs^2-16r^2s^2\right)$$

$$+\frac{(4R+r)^2+s^2}{2}+4\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)$$

$$= \frac{(4R+r)^2-s^2}{2}+\frac{(4R+r)^2+s^2}{2}+4(s^2-12Rr-3r^2) \overset{\text{Gerretsen}}{\ge}$$

$$16R^2+8Rr+r^2+4(16Rr-5r^2-12Rr-3r^2) = 16R^2+24Rr-31r^2$$

$$\Rightarrow \left(\sum_{cyc}\frac{n_a g_a}{h_a}\right)^2 \ge 16R^2+24Rr-31r^2 \therefore \frac{n_a g_a}{h_a}+\frac{n_b g_b}{h_b}+\frac{n_c g_c}{h_c} \ge$$

$$\sqrt{16R^2+24Rr-31r^2}\ \forall\,\Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$