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If $n_a, n_b, n_c \to$ Nagel cevians, $g_a, g_b, g_c \to$ Gergonne's cevians in Δ ABC, then

$$\frac{n_a g_a}{h_a^2}+\frac{n_b g_b}{h_b^2}+\frac{n_c g_c}{h_c^2}\ge\frac{\sqrt{R^2+3Rr-r^2}}{r}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$ and

$b^2(s-b)+c^2(s-c)=ag_a^2+a(s-b)(s-c)$ & adding the two, we get :

$$(b^2+c^2)(2s-b-c)=an_a^2+ag_a^2+2a(s-b)(s-c)\Rightarrow 2a(b^2+c^2)$$
$$=2a(n_a^2+g_a^2)+a(a+b-c)(c+a-b)\Rightarrow 2(b^2+c^2)$$
$$=2(n_a^2+g_a^2)+a^2-(b-c)^2\Rightarrow 2(b^2+c^2)-a^2+(b-c)^2=2(n_a^2+g_a^2)$$
$$\Rightarrow 4m_a^2+(b-c)^2=2(n_a^2+g_a^2)\Rightarrow 2(b-c)^2+4s(s-a)=2(n_a^2+g_a^2)$$
$$\Rightarrow n_a^2+g_a^2\overset{(*)}{=}(b-c)^2+2s(s-a)$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$$
$$=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$$
$$s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$
$$=as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$
$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2\overset{(**)}{=}s\left(s-a+\frac{(b-c)^2}{a}\right)$$

Via $(*)$ and $(**)$, $g_a^2=(b-c)^2+2s(s-a)-s^2+\frac{4s(s-b)(s-c)}{a}$

$$=s^2-2sa+a^2+(b-c)^2-a^2+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+(b-c+a)(b-c-a)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2-4(s-b)(s-c)+\frac{4s(s-b)(s-c)}{a}$$
$$=(s-a)^2+4(s-b)(s-c)\left(\frac{s}{a}-1\right)=(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}$$
$$=(s-a)\left(s-a+\frac{a^2-(b-c)^2}{a}\right)\Rightarrow g_a^2\overset{(***)}{=}(s-a)\left(s-\frac{(b-c)^2}{a}\right)$$
$$\therefore (**),(***)\Rightarrow n_a^2g_a^2=s(s-a)\left(s-a+\frac{(b-c)^2}{a}\right)\left(s-\frac{(b-c)^2}{a}\right)$$
$$=s(s-a)\left(s(s-a)+s.\frac{(b-c)^2}{a}-\frac{(b-c)^2}{a}(s-a)-\frac{(b-c)^4}{a^2}\right)$$

$$= s(s-a)\left(s(s-a)+(b-c)^2-\frac{(b-c)^4}{a^2}\right)$$

$$= s^2(s-a)^2+(b-c)^2.\frac{4s(s-a)(s-b)(s-c)}{a^2} = s^2(s-a)^2+(b-c)^2.\frac{4F^2}{a^2}$$

$$\Rightarrow n_a^2 g_a^2 = s^2(s-a)^2+(b-c)^2.h_a^2 \text{ and analogs} \rightarrow (1) \therefore n_a^2 g_a^2.n_b^2 g_b^2 =$$

$$\left(s^2(s-a)^2+(b-c)^2.h_a^2\right).\left(s^2(s-b)^2+(c-a)^2.h_b^2\right)$$

$$\geq \left(s^2(s-a)(s-b)+|b-c||c-a|.h_a h_b\right)^2$$

$$\therefore \boxed{n_a g_a.n_b g_b \geq s^2(s-a)(s-b)+|b-c||c-a|.h_a h_b} \text{ and analogs} \rightarrow (2)$$

$$\text{Now}, \sum_{cyc}\frac{n_a^2 g_a^2}{h_a^4} \overset{\text{via (1)}}{=} \sum_{cyc}\frac{a^4 s^2(s-a)^2}{16r^4s^4}+\sum_{cyc}\frac{a^2(b-c)^2}{4r^2s^2}$$

$$= \frac{1}{16r^4s^2}.\left(\left(\sum_{cyc}a^2(s-a)\right)^2-2\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right)\right)$$

$$+\frac{2\sum_{cyc}a^2b^2-16Rrs^2}{4r^2s^2}$$

$$= \frac{\left(2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\right)^2-2\left(-s^2((s^2+4Rr+r^2)^2-16Rrs^2)\right)}{16r^4s^2}$$

$$+\frac{4Rrs^2(s^2+4Rr+r^2)+(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{16r^4s^2}$$

$$+\frac{2s^2((s^2+4Rr+r^2)^2-16Rrs^2)-16Rrs^2}{4r^2s^2} \therefore \sum_{cyc}\frac{n_a^2 g_a^2}{h_a^4} =$$

$$\frac{3s^4+(8R^2-44Rr+14r^2)s^2-r^2(64R^3-16R^2r-20Rr^2-3r^3)}{8r^2s^2} \rightarrow (i)$$

$$\text{Again}, 2\sum_{cyc}\left(\frac{n_a g_a}{h_a^2}.\frac{n_b g_b}{h_b^2}\right) \overset{\text{via (2)}}{\geq} 2\sum_{cyc}\frac{s^2(s-a)(s-b)+|b-c||c-a|.h_a h_b}{h_a^2 h_b^2}$$

$$= \frac{\sum_{cyc}\left(b^2c^2(-s^2+sa+bc)\right)}{8r^4s^2}+\frac{1}{2r^2s^2}.\sum_{cyc}(ab.|b-c||c-a|)$$

$$\overset{\text{Triangle inequality}}{\geq} \frac{-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{8r^4s^2}+\frac{1}{2r^2s^2}.\left|\sum_{cyc}(ab(b-c)(c-a))\right|$$

$$= \frac{-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{8r^4s^2}$$

$$+\frac{1}{2r^2s^2}\cdot\left|abc\sum_{cyc}a-\sum_{cyc}a^2b^2-abc\sum_{cyc}a+abc\sum_{cyc}a\right|$$

$$=\frac{-s^2((s^2+4Rr+r^2)^2-16Rrs^2)+4Rrs^2(s^2+4Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)}{8r^4s^2}$$

$$+\frac{(s^2+4Rr+r^2)^2-16Rrs^2-8Rrs^2}{2r^2s^2}\left(\because\sum_{cyc}a^2b^2\geq abc\sum_{cyc}a\right)$$

$$\therefore 2\sum_{cyc}\left(\frac{n_a g_a}{h_a^2}\cdot\frac{n_b g_b}{h_b^2}\right)\geq$$

$$\frac{5s^4-(68Rr-10r^2)s^2+r^2(64R^3+112R^2r+44Rr^2+5r^3)}{8r^2s^2}\to(ii)$$

$$\therefore \text{via (i) and (ii)}, \sum_{cyc}\frac{n_a^2 g_a^2}{h_a^4}+2\sum_{cyc}\left(\frac{n_a g_a}{h_a^2}\cdot\frac{n_b g_b}{h_b^2}\right)\geq$$

$$\frac{3s^4+(8R^2-44Rr+14r^2)s^2-r^2(64R^3-16R^2r-20Rr^2-3r^3)}{8r^2s^2}$$

$$+\frac{5s^4-(68Rr-10r^2)s^2+r^2(64R^3+112R^2r+44Rr^2+5r^3)}{8r^2s^2}$$

$$=\frac{s^4+(R^2-14Rr+3r^2)s^2+r^2(4R+r)^2}{r^2s^2}\overset{?}{\geq}\frac{R^2+3Rr-r^2}{r^2}$$

$$\Leftrightarrow s^4-(17Rr-4r^2)s^2+r^2(4R+r)^2\underset{(\bullet)}{\overset{?}{\geq}}0 \text{ and } \because (s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0$$

$$\therefore \text{in order to prove } (\bullet), \text{it suffices to prove : LHS of } (\bullet)\geq(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (5R-2r)s^2\underset{(\bullet\bullet)}{\overset{?}{\geq}}r(80R^2-56Rr+8r^2)$$

$$\text{Now, } (5R-2r)s^2\overset{\text{Rouche}}{\geq}(5R-2r)\left(2R^2+10Rr-r^2-2(R-2r).\sqrt{R^2-2Rr}\right)$$

$$\overset{?}{\geq}r(80R^2-56Rr+8r^2)$$

$$\Leftrightarrow (R-2r)(10R^2-14Rr+3r^2)\underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}}2(R-2r)(5R-2r)\sqrt{R^2-2Rr}$$

$$\text{and } \because R-2r\overset{\text{Euler}}{\geq}0 \therefore \text{in order to prove } (\bullet\bullet\bullet), \text{it suffices to prove :}$$

$$(10R^2-14Rr+3r^2)^2>\left(2(5R-2r)\sqrt{R^2-2Rr}\right)^2\Leftrightarrow r^2(80R^2-52Rr+9r^2)>0$$

$$\Leftrightarrow r^2(54R^2+26R(R-2r)+9r^2)>0\to\text{true}\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(\bullet\bullet\bullet)\Rightarrow(\bullet\bullet)\Rightarrow(\bullet)$$

$$\text{is true} \therefore \left(\sum_{cyc}\frac{n_a g_a}{h_a^2}\right)^2 = \sum_{cyc}\frac{n_a^2 g_a^2}{h_a^4} + 2\sum_{cyc}\left(\frac{n_a g_a}{h_a^2}\cdot\frac{n_b g_b}{h_b^2}\right) \ge \frac{R^2+3Rr-r^2}{r^2}$$

$$\Rightarrow \frac{n_a g_a}{h_a^2} + \frac{n_b g_b}{h_b^2} + \frac{n_c g_c}{h_c^2} \ge \frac{\sqrt{R^2+3Rr-r^2}}{r}$$

$\forall\,\Delta\,ABC$, " = " iff $\Delta\,ABC$ is equilateral (QED)