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In any Δ ABC with $p_a, p_b, p_c \to$ Spieker cevians, $n_a, n_b, n_c \to$ Nagel cevians, the following relationship holds :

$$n_a + n_b + n_c \geq p_a + p_b + p_c + \frac{2}{3} \cdot \frac{a^2 + b^2 + c^2 - ab - bc - ca}{s}$$

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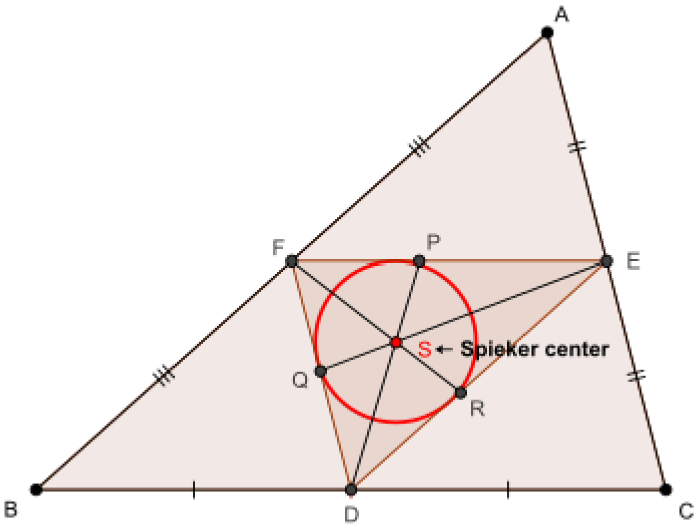

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now}, \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again}, \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$
$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$
$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now}, [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now}, b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

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$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Again, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$

$$\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2) \Rightarrow s(b^2+c^2-a^2-2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 +$$

$$s(2bc\cos A - 2bc) = as^2 - 4sbc\sin^2\frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 = s\left(s - \frac{a^2-(b-c)^2}{a}\right) \Rightarrow n_a^2 = s\left(s - a + \frac{(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2 \overset{(\bullet\bullet\bullet\bullet)}{=} s(s-a) + \frac{s}{a}.(b-c)^2$$

$$\text{Now, } n_a - p_a \overset{?}{\geq} \frac{(b-c)^2}{3a} \Leftrightarrow n_a \overset{?}{\geq} p_a + \frac{(b-c)^2}{3a}$$

$$\Leftrightarrow n_a^2 \overset{?}{\geq} p_a^2 + \frac{(b-c)^4}{9a^2} + \frac{2p_a}{3a}.(b-c)^2$$

$$\overset{\text{via } (\bullet\bullet\bullet) \text{ and } (\bullet\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a) + \frac{s}{a}.(b-c)^2 \overset{?}{\geq} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{9a^2}$$

$+\frac{2p_a}{3a}.(b-c)^2$ and $\because (b-c)^2 \geq 0 \therefore$ in order to prove this, it suffices to prove :

$\frac{s}{a} - \frac{s(3s+a)}{(2s+a)^2} \overset{?}{>} \frac{(b-c)^2}{9a^2} + \frac{2p_a}{3a}$ and $\because (b-c)^2 < a^2 \therefore$ in order to prove this,

it suffices to prove : $\frac{s}{a} - \frac{s(3s+a)}{(2s+a)^2} - \frac{1}{9} \overset{?}{>} \frac{2p_a}{3a} \Leftrightarrow \frac{36s^3+5s^2a-4sa^2-a^3}{9a(2s+a)^2} \overset{?}{>} \frac{2p_a}{3a}$

$$\overset{\text{via } (\bullet\bullet\bullet)}{\Leftrightarrow} \frac{(36s^3+5s^2a-4sa^2-a^3)^2}{81a^2(2s+a)^4} \overset{?}{>} \frac{4}{9a^2}\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\left(\because 36s^3+5s^2a-4sa^2-a^3 = 36s^3 + a(s-a)(5s+a) \overset{s>a}{>} 36s^3 > 0\right)$$

and $\because (b-c)^2 < a^2 \therefore$ in order to prove this, it suffices to prove :

$$\frac{(36s^3+5s^2a-4sa^2-a^3)^2}{81a^2(2s+a)^4} \overset{?}{>} \frac{4\left(s(s-a)(2s+a)^2 + sa^2(3s+a)\right)}{9a^2(2s+a)^2}$$

$$\Leftrightarrow 720t^6 - 216t^5 - 407t^4 - 112t^3 + 6t^2 + 8t + 1 \overset{?}{>} 0 \quad \left(t = \frac{s}{a}\right)$$

$$\Leftrightarrow (t-1)\begin{pmatrix} 720t^5 + 504t^4 + 72t^3 + 15t^2(t-1) + 9t(t+1)(t-1) \\ +(t-1)(t^2+t+1) \end{pmatrix} \overset{?}{>} 0$$

$\rightarrow$ true $\because t = \frac{s}{a} > 1 \therefore n_a - p_a \geq \frac{(b-c)^2}{3a} \overset{s>a}{\geq} \frac{(b-c)^2}{3s}$ and analogs

$$\Rightarrow \sum_{\text{cyc}} n_a - \sum_{\text{cyc}} p_a \geq \frac{2}{3s}\left(\sum_{\text{cyc}} a^2 - \sum_{\text{cyc}} ab\right)$$

$$\therefore n_a + n_b + n_c \geq p_a + p_b + p_c + \frac{2}{3}.\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

$\forall \Delta ABC$, with equality iff ΔABC is equilateral (QED)