

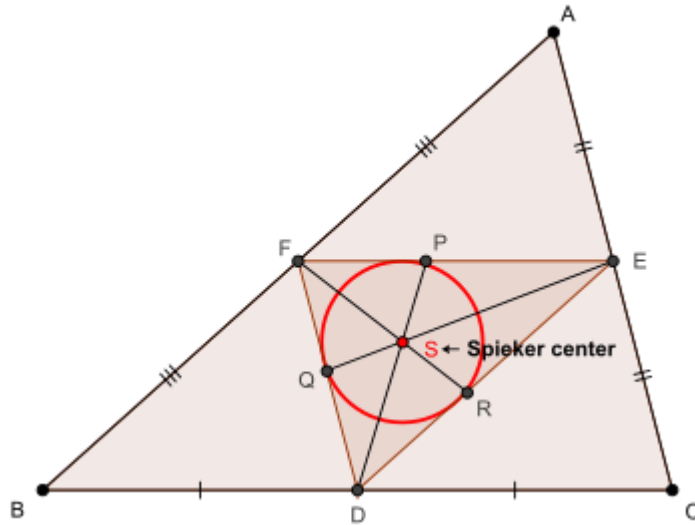
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In any ΔABC with $p_a, p_b, p_c \rightarrow$ Spieker cevians, $n_a, n_b, n_c \rightarrow$ Nagel cevians, the following relationship holds :

$$\frac{n_a - p_a}{h_a} + \frac{n_b - p_b}{h_b} + \frac{n_c - p_c}{h_c} \geq \frac{a^2 + b^2 + c^2 - ab - bc - ca}{3F}$$

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Let AS produced meet BC at X and $m(\sphericalangle BAX) = \alpha$ and $m(\sphericalangle CAX) = \beta$ (say) and inradius of $\Delta DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4}\right) \left(\frac{b^2}{4}\right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4\right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\begin{aligned} \because \text{ Spieker center is incenter of } \Delta DEF, \therefore m(\sphericalangle AFS) &= B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2} \\ &= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\sphericalangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on ΔAFS and ΔAES , we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} \\ &= \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2} \\ \Rightarrow 2AS^2 &\stackrel{(1)}{=} \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} + \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} \end{aligned}$$

$$\begin{aligned}
 & - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 \text{Now, } & \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & = \frac{r}{2} \left(4R\cos\frac{C}{2} \sin\frac{A-B}{2} + 4R\cos\frac{B}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left(2\sin\frac{A+B}{2} \sin\frac{A-B}{2} + 2\sin\frac{A+C}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2 \left(1 - 2\sin^2\frac{A}{2} \right) \right) \\
 & = 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 & = \frac{Rr}{8Rs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 & = \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a \cdot 2bcc\cos A}{8s} = \frac{bc \left((2s-a)\sin^2\frac{A}{2} - a \left(1 - 2\sin^2\frac{A}{2} \right) \right)}{2s} \\
 & = \frac{bc \left((2s+a)\sin^2\frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 & \Rightarrow - \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \\
 \text{Again, } & \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 & = \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} \\
 \text{(i), (*), (**)} & \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 & = \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s} \\
 & = \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s} \\
 \text{Via sine law on } \triangle AFS, & \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{4s}{(a+b)\sin\frac{C}{2}} \\
 \Rightarrow c\sin\alpha & \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b\sin\beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS} \\
 \text{Now, [BAX] + [BAX]} & = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs \\
 \text{via (***) and (***)} & \Rightarrow \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS \\
 \Rightarrow p_a^2 & \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s}
 \end{aligned}$$

$$\therefore p_a^2 \stackrel{(*)}{=} \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2))$$

$$\begin{aligned} \text{Now, } b^3 + c^3 - abc + a(4m_a^2) &= b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2) \\ &= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2) \\ &= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2) \\ &= (2s+a)(b^2 - bc + c^2) + a\left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2\right) \\ &= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a) \cdot \end{aligned}$$

$$\frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)((z+x) + (x+y) - 2(y+z))}{4}$$

$$\begin{aligned} &- \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y) \\ &= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4} \\ &= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{a(b-c)^2}{4} \\ &= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{(a+2s-2s)(b-c)^2}{4} \\ &= (2s+a) \left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2} \right) + \frac{s(b-c)^2}{2} \end{aligned}$$

$$\therefore b^3 + c^3 - abc + a(4m_a^2) \stackrel{(**)}{=} (2s+a) \left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\begin{aligned} \therefore (*), (**) \Rightarrow p_a^2 &= \frac{2s}{(2s+a)^2} \left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2} \right) \\ &= s(s-a) + (b-c)^2 \left(\left(\frac{s}{2s+a} \right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4} \right) \\ &= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left(\frac{s}{2s+a} + \frac{1}{2} \right)^2 \\ &= s(s-a) + \frac{(b-c)^2}{4} \left(\frac{(4s+a)^2}{(2s+a)^2} - 1 \right) \\ &\Rightarrow p_a^2 \stackrel{(***)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \end{aligned}$$

$$\begin{aligned} \text{Again, Stewart's theorem } &\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \\ \Rightarrow s(b^2 + c^2) - bc(2s-a) &= an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &= an_a^2 + a(as - s^2) \Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + \\ s(2bc \cos A - 2bc) &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} \\ &= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as \left(\frac{a^2 - (b-c)^2}{a} \right) \end{aligned}$$

$$\Rightarrow n_a^2 = s \left(s - \frac{a^2 - (b-c)^2}{a} \right) \Rightarrow n_a^2 = s \left(s - a + \frac{(b-c)^2}{a} \right)$$

$$\Rightarrow n_a^2 \stackrel{(\dots)}{=} s(s-a) + \frac{s}{a} \cdot (b-c)^2$$

$$\text{Now, } n_a - p_a \stackrel{?}{\geq} \frac{(b-c)^2}{3a} \Leftrightarrow n_a \stackrel{?}{\geq} p_a + \frac{(b-c)^2}{3a}$$

$$\Leftrightarrow n_a^2 \stackrel{?}{\geq} p_a^2 + \frac{(b-c)^4}{9a^2} + \frac{2p_a}{3a} \cdot (b-c)^2$$

$$\text{via } (\dots) \text{ and } (\dots) \Leftrightarrow s(s-a) + \frac{s}{a} \cdot (b-c)^2 \stackrel{?}{\geq} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{9a^2}$$

+ $\frac{2p_a}{3a} \cdot (b-c)^2$ and $\because (b-c)^2 \geq 0 \therefore$ in order to prove this, it suffices to prove :

$$\frac{s}{a} - \frac{s(3s+a)}{(2s+a)^2} \stackrel{?}{>} \frac{(b-c)^2}{9a^2} + \frac{2p_a}{3a} \text{ and } \because (b-c)^2 < a^2 \therefore \text{ in order to prove this,}$$

it suffices to prove : $\frac{s}{a} - \frac{s(3s+a)}{(2s+a)^2} - \frac{1}{9} \stackrel{?}{>} \frac{2p_a}{3a} \Leftrightarrow \frac{36s^3 + 5s^2a - 4sa^2 - a^3}{9a(2s+a)^2} \stackrel{?}{>} \frac{2p_a}{3a}$

$$\Leftrightarrow \frac{(36s^3 + 5s^2a - 4sa^2 - a^3)^2}{81a^2(2s+a)^4} \stackrel{?}{>} \frac{4}{9a^2} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} \right)$$

$$\left(\because 36s^3 + 5s^2a - 4sa^2 - a^3 = 36s^3 + a(s-a)(5s+a) \stackrel{s > a}{>} 36s^3 > 0 \right)$$

and $\because (b-c)^2 < a^2 \therefore$ in order to prove this, it suffices to prove :

$$\frac{(36s^3 + 5s^2a - 4sa^2 - a^3)^2}{81a^2(2s+a)^4} \stackrel{?}{>} \frac{4 \left(s(s-a)(2s+a)^2 + sa^2(3s+a) \right)}{9a^2(2s+a)^2}$$

$$\Leftrightarrow 720t^6 - 216t^5 - 407t^4 - 112t^3 + 6t^2 + 8t + 1 \stackrel{?}{>} 0 \left(t = \frac{s}{a} \right)$$

$$\Leftrightarrow (t-1) \left(\begin{array}{c} 720t^5 + 504t^4 + 72t^3 + 15t^2(t-1) + 9t(t+1)(t-1) \\ + (t-1)(t^2+t+1) \end{array} \right) \stackrel{?}{>} 0$$

$$\rightarrow \text{true } \because t = \frac{s}{a} > 1 \therefore n_a - p_a \geq \frac{(b-c)^2}{3a} \Rightarrow \frac{n_a - p_a}{h_a} \geq \frac{(b-c)^2}{3a \cdot \frac{2rs}{a}} = \frac{(b-c)^2}{6F}$$

$$\text{and analogs } \Rightarrow \sum_{\text{cyc}} \frac{n_a - p_a}{h_a} \geq \sum_{\text{cyc}} \frac{(b-c)^2}{6F} = \frac{2}{6F} \left(\sum_{\text{cyc}} a^2 - \sum_{\text{cyc}} ab \right)$$

$$\therefore \frac{n_a - p_a}{h_a} + \frac{n_b - p_b}{h_b} + \frac{n_c - p_c}{h_c} \geq \frac{a^2 + b^2 + c^2 - ab - bc - ca}{3F}$$

$\forall \Delta ABC$, with equality iff ΔABC is equilateral (QED)