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If p_a, p_b, p_c

$\to$ Spieker cevians in Δ ABC, then the following relationship holds :

$$3\sum_{cyc}\left((h_b+h_c)(p_a-w_a)\right)\ge 4s(h_a+h_b+h_c-9r)$$

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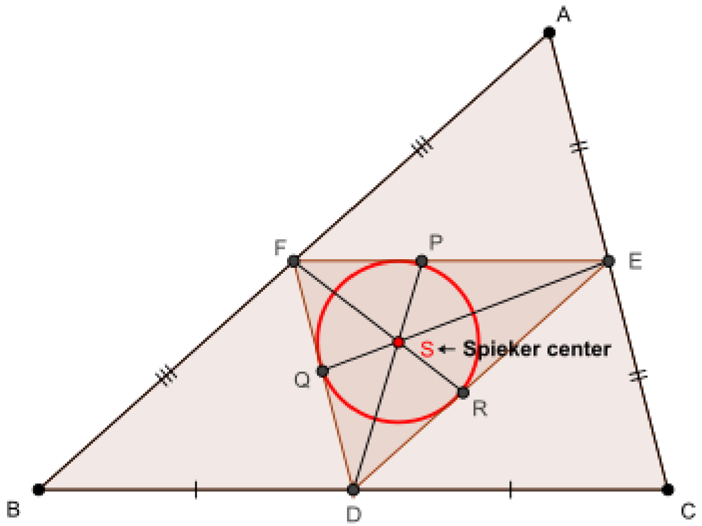

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$(say)

$$\text{Now, }16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of }\Delta\text{ DEF},\therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2}\text{ and }m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$
$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$
$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \overset{?}{\geq} w_a+\frac{2}{3}\cdot\frac{(b-c)^2}{b+c} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} s(s-a)$$

$$-\frac{s(s-a)(b-c)^2}{(2s-a)^2}+\frac{4}{9}\cdot\frac{(b-c)^4}{(b+c)^2}+\frac{4w_a}{3}\cdot\frac{(b-c)^2}{b+c}$$

$$\Leftrightarrow \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4}{9}\cdot\frac{(b-c)^2}{(b+c)^2} \boxed{\underset{(\blacksquare)}{\overset{?}{\geq}}} \frac{4w_a}{3(b+c)} \quad (\because (b-c)^2 \geq 0)$$

$$\text{We have: } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4}{9}\cdot\frac{(b-c)^2}{(b+c)^2} \overset{\because a^2>(b-c)^2}{>} \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}$$

$$-\frac{4}{9}\cdot\frac{a^2}{(2s-a)^2}=\frac{9\left(s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\right)-4a^2(2s+a)^2}{9(4s^2-a^2)^2}$$

$$=\frac{(s-a)\left(36s^3+18s^2a+5sa^2+a^3\right)}{9(4s^2-a^2)^2} \overset{s>a}{>} 0 \Rightarrow \text{LHS of } (\blacksquare)>0 \therefore (\blacksquare) \Leftrightarrow$$

$$\frac{\left(9T-4(b-c)^2(2s+a)^2\right)^2}{81(4s^2-a^2)^4} \overset{?}{\geq} \frac{16\left(s(s-a)(2s-a)^2-s(s-a)(b-c)^2\right)}{9(2s-a)^4}$$

$$\left(T=s(3s+a)(2s-a)^2+s(s-a)(2s+a)^2\right)$$

$$\Leftrightarrow \frac{81T^2+16(b-c)^4(2s+a)^4-72T(b-c)^2(2s+a)^2}{9(2s+a)^4} \overset{?}{\geq}$$

$$16s(s-a)(2s-a)^2-16s(s-a)(b-c)^2$$

$$\Leftrightarrow 16(2s+a)^4{\color{red}(b-c)^4}-\left(72T(2s+a)^2-144s(s-a)(2s+a)^4\right){\color{red}(b-c)^2}+81T^2$$

$$-144s(s-a)(2s-a)^2(2s+a)^4 \boxed{\underset{(\blacksquare\blacksquare)}{\overset{?}{\geq}}} 0$$

Now, LHS of $(\blacksquare\blacksquare)$ is a quadratic polynomial in $''(b-c)^2''$ with $\boxed{\text{discriminant}}=$

$$\left(72T(2s+a)^2-144s(s-a)(2s+a)^4\right)^2$$

$$-64(2s+a)^4\left(81T^2-144s(s-a)(2s-a)^2(2s+a)^4\right)$$

$$=72^2(2s+a)^4Q^2-$$

$$64.9(2s+a)^4\left(9T^2-16s(s-a)(2s-a)^2(2s+a)^4\right)$$

$${\color{blue}\left(Q=s(3s+a)(2s-a)^2-s(s-a)(2s+a)^2\right)}$$

$$=64.9(2s+a)^4\left(9Q^2-9T^2+16s(s-a)(2s-a)^2(2s+a)^4\right)$$

$$=-4s.64.9(2s+a)^4.a^7\left(176t^7-288t^6-40t^5+208t^4-13t^3-50t^2+3t+4\right)$$

$$\left(t=\frac{s}{a}\right)$$

$$=\boxed{-256s.9(2s+a)^4.a^7(t-1)^2\left(\begin{matrix}120t^5+\left(24t^5-24t^3\right)+\left(32t^5-32t^2\right)+\\ \left(64t^4-64t^3\right)+11t+4\end{matrix}\right)}<0$$

$$\because t>1 \Rightarrow \text{LHS of } (\blacksquare\blacksquare)>0 \Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true}$$

$$\therefore \boxed{\boxed{p_a \geq w_a+\frac{2}{3}\cdot\frac{(b-c)^2}{b+c}}} \text{ and analogs}$$

$$\Rightarrow 3\sum_{\text{cyc}}\left((h_b+h_c)(p_a-w_a)\right)-4s(h_a+h_b+h_c-9r)$$

$$= 3\sum_{cyc}\left(\left(\frac{ca+ab}{2R}\right)\left(\frac{2}{3}.\frac{(b-c)^2}{b+c}\right)\right) - 4s\left(\frac{s^2+4Rr+r^2}{2R} - 9r\right)$$

$$= \frac{1}{R}\sum_{cyc}\left(a\left(b^2+c^2-2bc\right)\right) - 4s\left(\frac{s^2-14Rr+r^2}{2R}\right)$$

$$= \frac{1}{R}\left(\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right) - 9abc\right) - 2s\left(\frac{s^2-14Rr+r^2}{R}\right)$$

$$= \frac{1}{R}.2s\left(s^2+4Rr+r^2-18Rr\right) - 2s\left(\frac{s^2-14Rr+r^2}{R}\right) = 0$$

$$\therefore 3\sum_{cyc}\left((h_b+h_c)(p_a-w_a)\right) \geq 4s(h_a+h_b+h_c-9r)\ \forall\ \Delta\ ABC,$$

with equality iff Δ ABC is equilateral (QED)