If p_a, p_b, p_c

→ Spieker cevians in Δ ABC, then the following relationship holds :

$$p_a + p_b + p_c \le w_a + w_b + w_c + \frac{4}{3}(\max\{a,b,c\} - \min\{a,b,c\})$$

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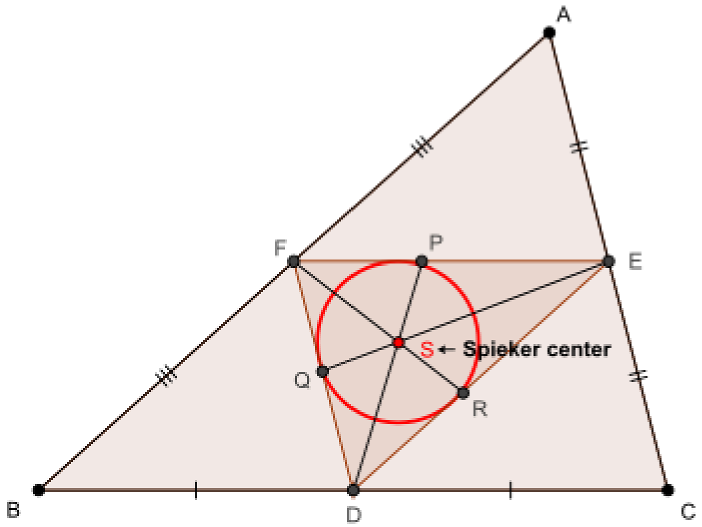

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF = r′ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a\left(4m_a^2\right)\right)}$$

$$\text{Now, } b^3+c^3-abc+a\left(4m_a^2\right) = b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)$$
$$= (b+c)\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2\right)+a\left(b^2+c^2-a^2\right)$$
$$= 2s\left(b^2-bc+c^2\right)+a\left(b^2-bc+c^2+bc-a^2\right)$$
$$= (2s+a)\left(b^2-bc+c^2\right)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(b^2-bc+c^2\right)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a\left(4m_a^2\right) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$
$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$
$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \overset{?}{\le} w_a+\frac{2}{3}|b-c| \Leftrightarrow p_a^2 \overset{?}{\le} w_a^2+\frac{4}{9}(b-c)^2+\frac{4}{3}.w_a.|b-c|$$
$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{(2s-a)^2}$$
$$-\frac{4}{9}(b-c)^2 \overset{?}{\le} \frac{4}{3}.w_a.|b-c|$$
$$\Leftrightarrow \left(\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{(2s-a)^2}-\frac{4}{9}\right)|b-c| \overset{?}{\le} \frac{4}{3}.w_a\ (\because |b-c|\ge 0)$$

$$\Leftrightarrow \frac{(20s^4 - 18s^3a - s^2a^2 - a^4)|b-c|}{9(4s^2-a^2)^2} \overset{?}{\leq} \frac{w_a}{3}$$

$$\Leftrightarrow \frac{(s-a)(20s^3 + 2s^2a + sa^2 + a^3)|b-c|}{3(4s^2-a^2)^2} \overset{?}{\leq} w_a$$

$$\Leftrightarrow \frac{(s-a)^2(20s^3 + 2s^2a + sa^2 + a^3)^2(b-c)^2}{9(4s^2-a^2)^4} \overset{?}{\leq} s(s-a) - \frac{s(s-a)(b-c)^2}{(2s-a)^2}$$

$$\Leftrightarrow \frac{(s-a)(20s^3 + 2s^2a + sa^2 + a^3)^2 + 9s(2s-a)^2(2s+a)^4}{9(4s^2-a^2)^4}.(b-c)^2 \overset{?}{\leq} s \text{ and}$$

$\because (b-c)^2 < a^2 \therefore$ in order to prove this, it suffices to prove :

$$9s(4s^2-a^2)^4 \overset{?}{>} a^2(s-a)(20s^3 + 2s^2a + sa^2 + a^3)^2 + 9sa^2(2s-a)^2(2s+a)^4$$

$$\Leftrightarrow 2304t^9 - 3280t^7 - 256t^6 + 1044t^5 + 288t^4 - 69t^3 - 33t^2 + t + 1 \overset{?}{>} 0$$

$$\left(t = \frac{s}{a}\right) \Leftrightarrow (t-1)\begin{pmatrix} 2212t^8 + 92(t^8 - t^6) + 884(t^7 - t^6) + 1232(t^7 - t^5) + \\ 188(t^7 - t^4) + 100t^3 + 28t^2 + 2(t^2 - t) + t^2 - 1 \end{pmatrix} \overset{?}{>} 0$$

$$\rightarrow \text{true} \because t = \frac{s}{a} > 1 \therefore p_a \leq w_a + \frac{2}{3}|b-c| \text{ and analogs}$$

$$\Rightarrow \boxed{p_a + p_b + p_c \leq w_a + w_b + w_c + \frac{2}{3}(|a-b| + |b-c| + |c-a|)} \rightarrow (m)$$

Now, we shall prove that : $\frac{1}{2}(|b-c| + |c-a| + |a-b|)$
$= max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (1)}}$ $a \geq b \geq c \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(b - c + a - c + a - b)$
$= a - c = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (2)}}$ $a \geq c \geq b \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(c - b + a - c + a - b)$
$= a - b = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (3)}}$ $b \geq c \geq a \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(b - c + c - a + b - a)$
$= b - a = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (4)}}$ $b \geq a \geq c \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(b - c + a - c + b - a)$
$= b - c = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (5)}}$ $c \geq a \geq b \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(c - b + c - a + a - b)$
$= c - b = max\{a,b,c\} - \min\{a,b,c\}$

$\boxed{\text{Case (6)}}$ $c \geq b \geq a \therefore \frac{1}{2}(|b-c| + |c-a| + |a-b|) = \frac{1}{2}(c - b + c - a + b - a)$
$= c - a = max\{a,b,c\} - \min\{a,b,c\} \therefore$ combining all 6 cases, we conclude :

$$\boxed{\frac{1}{2}(|b-c| + |c-a| + |a-b|) = max\{a,b,c\} - \min\{a,b,c\}} \rightarrow (n) \therefore (m) \text{ and } (n)$$

$$\Rightarrow p_a + p_b + p_c \leq w_a + w_b + w_c + \frac{4}{3}(max\{a,b,c\} - \min\{a,b,c\}) \ \forall \Delta ABC,$$

with equality iff Δ ABC is equilateral (QED)