

ROMANIAN MATHEMATICAL MAGAZINE

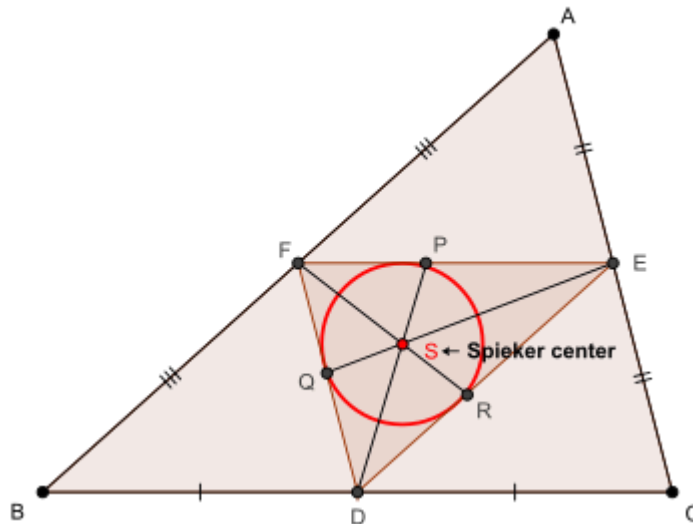
If p_a, p_b, p_c

→ Spieker cevians in $\triangle ABC$, then the following relationship holds :

$$p_a + p_b + p_c + 12r \geq \frac{7}{3}(h_a + h_b + h_c)$$

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Let AS produced meet BC at X and $m(\sphericalangle BAX) = \alpha$ and $m(\sphericalangle CAX) = \beta$ (say)
and inradius of $\triangle DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4}\right) \left(\frac{b^2}{4}\right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4\right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\begin{aligned} \because \text{ Spieker center is incenter of } \triangle DEF, \therefore m(\sphericalangle AFS) &= B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2} \\ &= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\sphericalangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on $\triangle AFS$ and $\triangle AES$, we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} \\ &= \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2} \end{aligned}$$

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$$\Rightarrow 2AS^2 \stackrel{(i)}{=} \frac{r^2}{4\sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} + \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2}$$

$$\text{Now, } \left(\frac{2r}{2\sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} + \left(\frac{2r}{2\sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2}$$

$$= \frac{r}{2} \left(4R \cos \frac{C}{2} \sin \frac{A-B}{2} + 4R \cos \frac{B}{2} \sin \frac{A-C}{2} \right)$$

$$= Rr \left(2 \sin \frac{A+B}{2} \sin \frac{A-B}{2} + 2 \sin \frac{A+C}{2} \sin \frac{A-C}{2} \right)$$

$$= Rr \left(1 - 2 \sin^2 \frac{B}{2} + 1 - 2 \sin^2 \frac{C}{2} - 2 \left(1 - 2 \sin^2 \frac{A}{2} \right) \right)$$

$$= 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right)$$

$$= \frac{Rr}{8Rrs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2)$$

$$= \frac{4(b+c)bc \sin^2 \frac{A}{2} - 2a \cdot 2bcc \cos A}{8s} = \frac{bc \left((2s-a) \sin^2 \frac{A}{2} - a \left(1 - 2 \sin^2 \frac{A}{2} \right) \right)}{2s}$$

$$= \frac{bc \left((2s+a) \sin^2 \frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow - \left(\frac{2r}{2\sin \frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin \frac{A-B}{2} - \left(\frac{2r}{2\sin \frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin \frac{A-C}{2}$$

$$\stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again, } \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{r^2}{4\sin^2 \frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right)$$

$$= \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2 \frac{B}{2}} + \frac{r^2}{4\sin^2 \frac{C}{2}}$$

$$(i), (*), (**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{4s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \stackrel{(ii)}{\Rightarrow} 2AS^2 = \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \triangle AFS, \frac{r}{2\sin \frac{C}{2} \sin \alpha} = \frac{AS}{\cos \frac{A-B}{2}} = \frac{cAS}{(a+b)\sin \frac{C}{2}}$$

$$\Rightarrow c \sin \alpha \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b \sin \beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS}$$

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$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2} p_a c \sin \alpha + \frac{1}{2} p_a b \sin \beta = rs$$

$$\text{via (***) and (***)} \Rightarrow \frac{p_a(a+b+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS$$

$$\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s}$$

$$\therefore p_a^2 \stackrel{(\bullet)}{=} \frac{2s}{(2s+a)^2} (b^3 + c^3 - abc + a(4m_a^2))$$

$$\text{Now, } b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 - abc + a(2b^2 + 2c^2 - a^2)$$

$$= (b+c)(b^2 - bc + c^2) + a(b^2 - bc + c^2) + a(b^2 + c^2 - a^2)$$

$$= 2s(b^2 - bc + c^2) + a(b^2 - bc + c^2 + bc - a^2)$$

$$= (2s+a)(b^2 - bc + c^2) + a \left(\frac{(b+c)^2 - (b-c)^2}{4} - a^2 \right)$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2 - bc + c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \cdot \frac{4b^2 + 4c^2 - 4bc + a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2 + 4(x+y)^2 - 4(z+x)(x+y) + (y+z)((z+x) + (x+y) - 2(y+z))}{4}$$

$$- \frac{a(b-c)^2}{4} \quad (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a) \cdot \frac{4x(x+y+z) + 2x(y+z) + 3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{3}{4}(b-c)^2 + \frac{a(s-a)}{2} \right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a) \left(s(s-a) + \frac{(b-c)^2}{2} + \frac{a(s-a)}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore b^3 + c^3 - abc + a(4m_a^2) \stackrel{(\bullet\bullet)}{=} (2s+a) \left(\frac{(s-a)(2s+a)}{2} + \frac{(b-c)^2}{2} \right) + \frac{s(b-c)^2}{2}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2} \left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2} \right)$$

$$= s(s-a) + (b-c)^2 \left(\left(\frac{s}{2s+a} \right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4} \right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2 \cdot \left(\frac{s}{2s+a} + \frac{1}{2} \right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4} \left(\frac{(4s+a)^2}{(2s+a)^2} - 1 \right)$$

$$\Rightarrow p_a^2 \stackrel{(\dots)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \stackrel{?}{\geq} \frac{(2b^2 - bc + 2c^2)^2}{36R^2 \cdot \frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \stackrel{?}{\geq} \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1 \right) \left(\frac{2b^2 - bc + 2c^2}{3bc} + 1 \right)$$

$$\stackrel{\text{via } (\dots)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \stackrel{?}{\geq} \frac{4(b-c)^2(b^2 + bc + c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2 h_a^2 (2s+a)^2} \stackrel{?}{\geq} \frac{4(b-c)^2(b^2 + bc + c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{4s(s-a)(s-b)(s-c)(2s+a)^2} \stackrel{?}{\geq} \frac{b^2 + bc + c^2}{9b^2c^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \stackrel{?}{\geq} (b^2 + bc + c^2)(s-b)(s-c)$$

$$= (b^2 + bc + c^2)(-s(s-a) + bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \stackrel{?}{\geq} (bc - s(s-a))(b^2 + c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \stackrel{?}{\geq} (bc - s(s-a))((2s-a)^2 - 2bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \stackrel{?}{\geq} (bc - s(s-a))(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc(s(s-a) + (2s-a)^2) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2} \cdot b^2c^2 - (5s^2 - 5sa + a^2) \cdot bc + s(s-a)(2s-a)^2 \stackrel{?}{\geq} 0} \quad (\blacksquare)$$

Now, LHS of (\blacksquare) is a quadratic polynomial with discriminant =

$$\begin{aligned} & (5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2} \cdot s(2s-a)^2 \\ &= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2} \\ &= \frac{-a^2(s-a)((s-a)(12s^2 + 6sa + 5a^2) + 6a^3)}{(2s+a)^2} < 0 \quad (\because s > a) \end{aligned}$$

$\therefore (\blacksquare)$ is true (strict inequality) $\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R}$ and analogs

$$\begin{aligned} \Rightarrow p_a + p_b + p_c + 12r - \frac{7}{3}(h_a + h_b + h_c) &\geq \sum_{\text{cyc}} \frac{2b^2 - bc + 2c^2}{6R} - \frac{7}{6R} \sum_{\text{cyc}} bc + 12r \\ &= \frac{4 \sum_{\text{cyc}} a^2 - 8 \sum_{\text{cyc}} bc + 72Rr}{6R} = \frac{8(s^2 - 4Rr - r^2) - 8(s^2 + 4Rr + r^2) + 72Rr}{6R} \end{aligned}$$

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$$= \frac{8r(R - 2r)}{6R} \stackrel{\text{Euler}}{\geq} 0 \therefore p_a + p_b + p_c + 12r \geq \frac{7}{3}(h_a + h_b + h_c)$$

$\forall \Delta ABC$, with equality iff ΔABC is equilateral (QED)