If p_a, p_b, p_c

→ Spieker cevians in Δ ABC, then the following relationship holds :

$$p_a + p_b + p_c \geq h_a + h_b + h_c + \frac{4}{3}\cdot\frac{a^2 + b^2 + c^2 - ab - bc - ca}{s}$$

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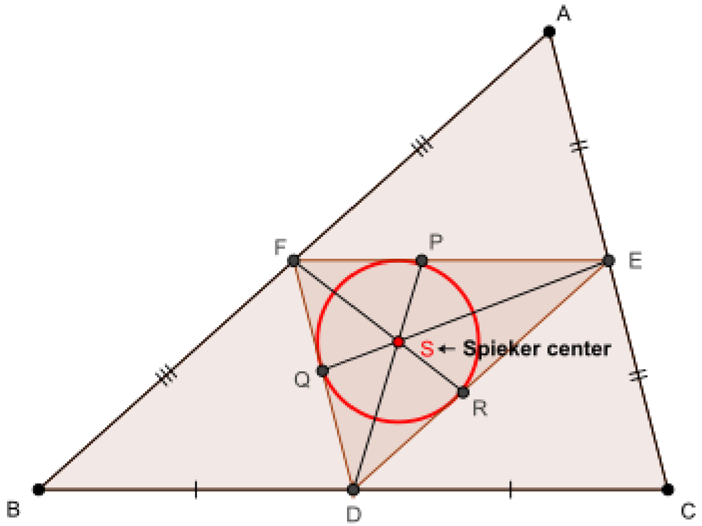

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF = r′(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a\left(4m_a^2\right)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \overset{?}{\geq} h_a+\frac{2}{3}.\frac{(b-c)^2}{a} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\overset{?}{\geq} s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}+\frac{4}{9}.\frac{(b-c)^4}{a^2}+\frac{4h_a}{3}.\frac{(b-c)^2}{a}$$

$$\Leftrightarrow \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2} \boxed{\overset{?}{\underset{(\blacksquare)}{\geq}}} \frac{4h_a}{3a}\ (\because (b-c)^2 \geq 0)$$

We have : $\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2}\overset{\because a^2>(b-c)^2}{>}\frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4}{9}$

$$=\frac{9s(3s+a)a^2+9s(s-a)(2s+a)^2-4a^2(2s+a)^2}{9a^2(2s+a)^2}$$

$$=\frac{4(s-a)\left(9s^3+9s^2a+5sa^2+a^3\right)}{9a^2(2s+a)^2}\overset{s>a}{>}0\Rightarrow \text{LHS of } (\blacksquare)>0 \therefore (\blacksquare)\Leftrightarrow$$

$$\frac{T^2}{a^4(2s+a)^4}+\frac{16(b-c)^4}{81a^4}-\frac{8T(b-c)^2}{9a^4(2s+a)^2}\overset{?}{\geq}\frac{16}{9a^2}.\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$\left(T=s(3s+a)a^2+s(s-a)(2s+a)^2\right)$$

$$\Leftrightarrow\frac{16(b-c)^4}{81a^4}-\frac{8(b-c)^2}{9a^4}\left(\frac{T}{(2s+a)^2}-2s(s-a)\right)+\frac{T^2}{a^4(2s+a)^4}-\frac{16s(s-a)}{9a^2}$$

$$\Leftrightarrow\left(\frac{4(b-c)^2}{9}\right)^2+\frac{4(b-c)^2}{9}.\frac{4s\left(2s^3-3sa^2-a^3\right)}{(2s+a)^2}$$

$$+\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}\boxed{\overset{?}{\underset{(\blacksquare\blacksquare)}{\geq}}}0$$

Now, LHS of $(\blacksquare\blacksquare)$ is a quadratic polynomial in $''\frac{4(b-c)^2}{9}''$ whose $\boxed{\text{discriminant}}$

$$=\frac{16s^2\left(2s^3-3sa^2-a^3\right)^2}{(2s+a)^4}-4.\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}$$

$$=-\frac{16sa^7}{9(2s+a)^4}.\left(44t^5-28t^4-49t^3+10t^2+19t+4\right)\ \left(t=\frac{s}{a}\right)$$

$$=-\frac{16sa^7}{9(2s+a)^4}.(t-1)^2\left(44t^3+60t^2+27t+4\right)<0\Rightarrow \text{LHS of } (\blacksquare\blacksquare)>0$$

$$\Rightarrow(\blacksquare\blacksquare)\Rightarrow(\blacksquare)\text{ is true }\therefore p_a\geq h_a+\frac{2}{3}.\frac{(b-c)^2}{a}\overset{s>a}{\geq}h_a+\frac{2}{3}.\frac{(b-c)^2}{s}\text{ and analogs}$$

$$\Rightarrow\sum_{cyc}p_a\geq\sum_{cyc}h_a+\frac{2}{3s}.\sum_{cyc}(b-c)^2$$

$$\therefore p_a+p_b+p_c\geq h_a+h_b+h_c+\frac{4}{3}.\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

$\forall\ \Delta$ ABC, with equality iff Δ ABC is equilateral (QED)