ROMANIAN MATHEMATICAL MAGAZINE

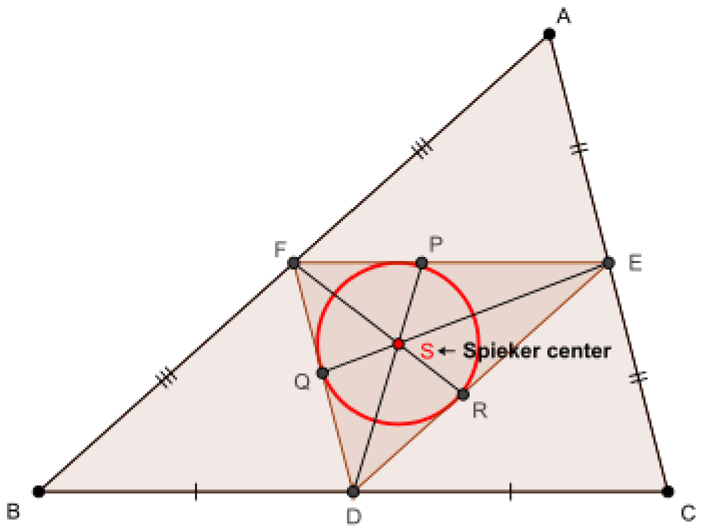
In any Δ ABC with

$p_a \rightarrow$ Spieker cevian, the following relationship holds :

$$p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we $arrive$ at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\ \text{and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a).$$
$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\big((z+x)+(x+y)-2(y+z)\big)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$
$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore\ \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2} + \frac{(2s+a)(b-c)^2}{2} + \frac{s(b-c)^2}{2}\right)$$

$$= s(s-a) + (b-c)^2\left(\left(\frac{s}{2s+a}\right)^2 + \frac{s}{2s+a} + \frac{1}{4} - \frac{1}{4}\right)$$

$$= s(s-a) - \frac{(b-c)^2}{4} + (b-c)^2.\left(\frac{s}{2s+a} + \frac{1}{2}\right)^2$$

$$= s(s-a) + \frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2} - 1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2 - bc + 2c^2)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2 - h_a^2}{h_a^2} \geq \left(\frac{2b^2 - bc + 2c^2}{3bc} - 1\right)\left(\frac{2b^2 - bc + 2c^2}{3bc} + 1\right)$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \big(bc - s(s-a)\big)(b^2+c^2) + b^2c^2 - bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq \big(bc - s(s-a)\big)\left((2s-a)^2 - 2bc\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} - b^2c^2 + bcs(s-a) \geq$$

$$\big(bc - s(s-a)\big)(2s-a)^2 - 2b^2c^2 + 2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} + b^2c^2 + s(s-a)(2s-a)^2 - bc\left(s(s-a) + (2s-a)^2\right) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3 - 12sa^2 - 4a^3}{4(s-a)(2s+a)^2}.b^2c^2 - (5s^2 - 5sa + a^2).bc + s(s-a)(2s-a)^2 \overset{(?)}{\geq} 0}$$

Now, LHS of (?) is a quadratic polynomial with discriminant =

$$(5s^2 - 5sa + a^2)^2 - \frac{25s^3 - 12sa^2 - 4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2(12s^4 - 18s^3a + 5s^2a^2 + 2sa^3 - a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)(12s^2 + 6sa + 5a^2) + 6a^3\right)}{(2s+a)^2} < 0\ (\because s > a)$$

$$\therefore (?)\text{ is true (strict inequality)}$$

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R}\ \forall \Delta ABC \to (m)$$

$$\text{Again, } p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2 \Leftrightarrow \frac{p_a^2 - h_a^2}{p_a + h_a} \leq \frac{16R}{9} \cdot \frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{p_a + h_a} \leq \frac{16R}{9} \cdot \frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s^4}{(2s+a)^2} \overset{(\blacksquare\blacksquare)}{\leq} \frac{4R}{9} \cdot (p_a + h_a)\ (\because (b-c)^2 \geq 0)$$

$$\text{Now, via (m), } \frac{4R}{9} \cdot (p_a + h_a) \geq \frac{4R}{9} \cdot \frac{2b^2 - bc + 2c^2 + 3bc}{6R} = \frac{4}{27} \cdot (b^2 + bc + c^2) \geq$$

$$\frac{4}{27} \cdot \frac{3}{4} \cdot (2s-a)^2 \overset{?}{>} \frac{s^4}{(2s+a)^2} \Leftrightarrow 4s^2 - a^2 \overset{?}{>} 3s^2 \Leftrightarrow s^2 \overset{?}{>} a^2 \to \text{true} \Rightarrow (\blacksquare\blacksquare)$$

$$\text{is true } \therefore p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2\ \forall \Delta ABC, '' ='' \text{ iff } b = c\ (QED)$$