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In any Δ ABC with $p_a \to$ Spieker cevian, the following relationship holds :

$$p_a \le h_a + \frac{64}{27}(R - 2r)$$

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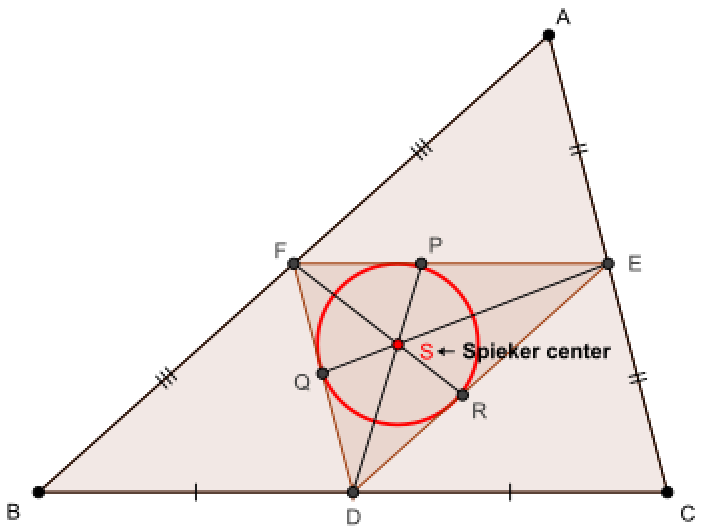

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX] + [BAX] = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$= (b+c)(b^2-bc+c^2) + a(b^2-bc+c^2) + a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2) + a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2) + a\left(\frac{(b+c)^2-(b-c)^2}{4} - a^2\right)$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(b+c+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2) + \frac{a(2s-a+2a)(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a = y+z, b = z+x, c = x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4} - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right) - \frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right) + \frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet), (\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \geq \frac{2b^2-bc+2c^2}{6R} \Leftrightarrow \frac{p_a^2}{h_a^2} \geq \frac{(2b^2-bc+2c^2)^2}{36R^2.\frac{b^2c^2}{4R^2}}$$

$$\Leftrightarrow \frac{p_a^2-h_a^2}{h_a^2} \geq \left(\frac{2b^2-bc+2c^2}{3bc}-1\right)\left(\frac{2b^2-bc+2c^2}{3bc}+1\right)$$

$$\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} \frac{s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-s(s-a)+\frac{s(s-a)(b-c)^2}{a^2}}{h_a^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{4s^4(b-c)^2}{a^2h_a^2(2s+a)^2} \geq \frac{4(b-c)^2(b^2+bc+c^2)}{9b^2c^2}$$

$$\Leftrightarrow \frac{s^4}{4s(s-a)(s-b)(s-c)(2s+a)^2} \geq \frac{b^2+bc+c^2}{9b^2c^2} \quad (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq (b^2+bc+c^2)(s-b)(s-c)$$

$$= (b^2+bc+c^2)(-s(s-a)+bc)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2} \geq \left(bc-s(s-a)\right)(b^2+c^2)+b^2c^2-bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \geq \left(bc-s(s-a)\right)\left((2s-a)^2-2bc\right)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}-b^2c^2+bcs(s-a) \geq$$
$$\left(bc-s(s-a)\right)(2s-a)^2-2b^2c^2+2bcs(s-a)$$

$$\Leftrightarrow \frac{9s^3b^2c^2}{4(s-a)(2s+a)^2}+b^2c^2+s(s-a)(2s-a)^2-bc\left(s(s-a)+(2s-a)^2\right) \geq 0$$

$$\Leftrightarrow \boxed{\frac{25s^3-12sa^2-4a^3}{4(s-a)(2s+a)^2}.b^2c^2-(5s^2-5sa+a^2).bc+s(s-a)(2s-a)^2 \overset{(\square)}{\geq} 0}$$

Now, LHS of (□) is a quadratic polynomial with discriminant =

$$(5s^2-5sa+a^2)^2-\frac{25s^3-12sa^2-4a^3}{(2s+a)^2}.s(2s-a)^2$$

$$= \frac{-a^2(12s^4-18s^3a+5s^2a^2+2sa^3-a^4)}{(2s+a)^2}$$

$$= \frac{-a^2(s-a)\left((s-a)(12s^2+6sa+5a^2)+6a^3\right)}{(2s+a)^2} < 0 \ (\because s > a)$$

$$\therefore (?) \text{ is true (strict inequality)}$$

$$\therefore p_a \geq \frac{2b^2 - bc + 2c^2}{6R} \ \forall \Delta ABC \to (m)$$

$$\text{Again, } p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2 \Leftrightarrow \frac{p_a^2 - h_a^2}{p_a + h_a} \leq \frac{16R}{9} \cdot \frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2} - s(s-a) + \frac{s(s-a)(b-c)^2}{a^2}}{p_a + h_a} \leq \frac{16R}{9} \cdot \frac{(b-c)^2}{a^2}$$

$$\Leftrightarrow \frac{s^4}{(2s+a)^2} \overset{(\blacksquare\blacksquare)}{\leq} \frac{4R}{9} \cdot (p_a + h_a) \ (\because (b-c)^2 \geq 0)$$

$$\text{Now, via (m), } \frac{4R}{9} \cdot (p_a + h_a) \geq \frac{4R}{9} \cdot \frac{2b^2 - bc + 2c^2 + 3bc}{6R} = \frac{4}{27} \cdot (b^2 + bc + c^2) \geq$$

$$\frac{4}{27} \cdot \frac{3}{4} \cdot (2s-a)^2 \overset{?}{>} \frac{s^4}{(2s+a)^2} \Leftrightarrow 4s^2 - a^2 \overset{?}{>} 3s^2 \Leftrightarrow s^2 \overset{?}{>} a^2 \to \text{true} \Rightarrow (\blacksquare\blacksquare)$$

$$\text{is true} \therefore p_a \leq h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2 \overset{?}{\leq} h_a + \frac{64}{27}(R - 2r)$$

$$\Leftrightarrow \frac{4}{3}(R-2r) \cdot a^2 \overset{?}{\geq} R(b-c)^2 \Leftrightarrow$$

$$\frac{4}{3} \cdot R\left(1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\right) \cdot 16R^2 \sin^2\frac{A}{2}\cos^2\frac{A}{2}$$

$$\overset{?}{\geq} R \cdot 16R^2 \sin^2\frac{A}{2}\sin^2\frac{B-C}{2}$$

$$\Leftrightarrow \frac{4}{3}\left(1 - 4\sin\frac{A}{2}\cos\frac{B-C}{2} + 4\sin^2\frac{A}{2}\right)\left(1 - \sin^2\frac{A}{2}\right) \overset{?}{\geq} 1 - \cos^2\frac{B-C}{2}$$

$$\Leftrightarrow 3\cos^2\frac{B-C}{2} - 16(x - x^3)\cos\frac{B-C}{2} + 1 + 12x^2 - 16x^4 \ \boxed{\underset{(\blacksquare\blacksquare\blacksquare)}{\overset{?}{\geq}}} \ 0 \ \left(x = \sin\frac{A}{2}\right)$$

Now, LHS of $(\blacksquare\blacksquare\blacksquare)$ is a quadratic polynomial in $\cos\frac{B-C}{2}$ with discriminant $=$

$$256(x - x^3)^2 - 12(1 + 12x^2 - 16x^4) = 256x^6 - 320x^4 + 112x^2 - 12$$

$= 4(4x^2 - 1)^2(4x^2 - 3) \leq 0$ iff $x \leq \frac{\sqrt{3}}{2}$ and so, when $x \leq \frac{\sqrt{3}}{2}$, discriminant ≤ 0

$\Rightarrow$ LHS of $(\blacksquare\blacksquare\blacksquare) \geq 0 \Rightarrow (\blacksquare\blacksquare\blacksquare)$ is true and we now focus on the scenario when :

when $x > \frac{\sqrt{3}}{2}$ and then, in order to prove $(\blacksquare\blacksquare\blacksquare)$, it suffices to prove :

$$\cos\frac{B-C}{2} > \frac{8(x - x^3) + (4x^2 - 1) \cdot \sqrt{4x^2 - 3}}{3} \left(\because x > \frac{\sqrt{3}}{2} > \frac{1}{2} \Rightarrow 4x^2 - 1 > 0\right)$$

$$\text{and} \because \cos\frac{B-C}{2} = \frac{b+c}{a}\sin\frac{A}{2} \overset{\frac{b+c}{a} > 1}{>} x \ \therefore \text{it suffices to prove :}$$

$$x > \frac{8(x - x^3) + (4x^2 - 1).\sqrt{4x^2 - 3}}{3} \Leftrightarrow 8x^3 - 5x > (4x^2 - 1).\sqrt{4x^2 - 3}$$

$$\Leftrightarrow (8x^3 - 5x)^2 > (4x^2 - 3)(4x^2 - 1)^2 \left(\because x > \frac{\sqrt{3}}{2} \Rightarrow 8x^2 > 6 > 5 \Rightarrow 8x^3 > 5x\right)$$

$$\Leftrightarrow 3 - 3x^2 > 0 \Rightarrow 1 > x^2 \to \text{true} \Rightarrow (\blacksquare\blacksquare\blacksquare) \text{ is true and combining both cases,}$$

$$(\blacksquare\blacksquare\blacksquare) \text{ is true } \forall \Delta \, ABC \therefore p_a \le h_a + \frac{64}{27}(R - 2r),$$

$$'' ='' \text{ iff } \sin\frac{A}{2} = \frac{1}{2} \text{ and } B = C \Rightarrow '' ='' \text{ iff } \Delta \, ABC \text{ is equilateral (QED)}$$