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In any Δ ABC with $p_a \to$ Spieker cevian, $n_a \to$ Nagel cevians, the following relationships hold : $\frac{s(b-c)^2}{a(2s+a)} \le n_a - p_a \le \frac{s|b-c|}{2s+a}$

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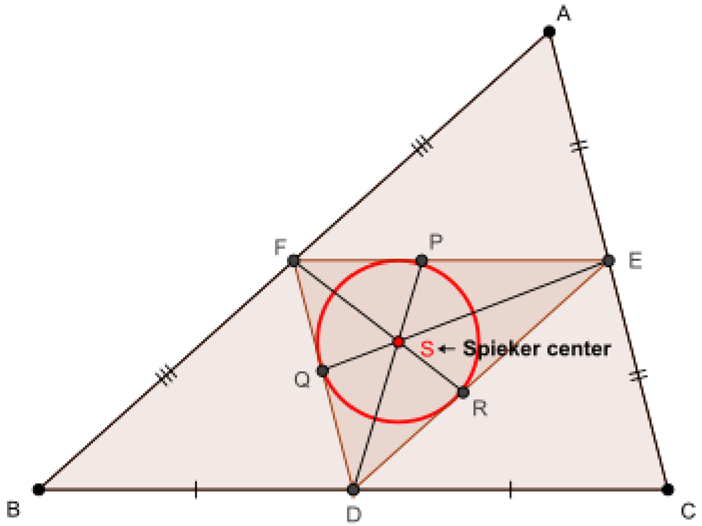

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$(say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via (***) and (****)}}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\Rightarrow p_a^2 - m_a^2 = \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right) - m_a^2$$

$$= \frac{2s}{(2s+a)^2}(b^3+c^3-abc) - \left(1-\frac{8sa}{(2s+a)^2}\right)m_a^2$$

$$= \frac{4(a+b+c)(b^3+c^3-abc)-(2b^2+2c^2-a^2)(b+c)^2}{4(2s+a)^2}$$

$$= \frac{a^2(b-c)^2+4a(b+c)(b-c)^2+2(b^2-c^2)^2}{4(2s+a)^2}$$

$$= \frac{(b-c)^2}{4(2s+a)^2}\left((a^2+2a(b+c)+(b+c)^2)+((b+c)^2+2a(b+c)+a^2)-a^2\right)$$

$$= \frac{(b-c)^2}{4(2s+a)^2}(2(a+b+c)^2-a^2) = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2}$$

$$\therefore p_a^2 - m_a^2 = \frac{(b-c)^2(8s^2-a^2)}{4(2s+a)^2} \ge 0 \Rightarrow p_a \overset{(\bullet\bullet)}{\ge} m_a$$

Now, $b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2\overset{(\bullet\bullet\bullet\bullet)}{=}s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

Again, Stewart's theorem $\Rightarrow b^2(s-c)+c^2(s-b)=an_a^2+a(s-b)(s-c)$ $\Rightarrow s(b^2+c^2)-bc(2s-a)=an_a^2+a(s^2-s(2s-a)+bc)\Rightarrow s(b^2+c^2)-2sbc$ $=an_a^2+a(as-s^2)\Rightarrow s(b^2+c^2-a^2-2bc)=an_a^2-as^2\Rightarrow an_a^2=as^2+$ $s(2bc\cos A-2bc)=as^2-4sbc\sin^2\frac{A}{2}=as^2-\frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$

$$= as^2-\frac{as(c+a-b)(a+b-c)}{a}=as^2-as\left(\frac{a^2-(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2=s\left(s-\frac{a^2-(b-c)^2}{a}\right)\Rightarrow n_a^2=s\left(s-a+\frac{(b-c)^2}{a}\right)$$

$$\Rightarrow n_a^2\overset{(\bullet\bullet\bullet\bullet\bullet)}{=}s(s-a)+\frac{s}{a}.(b-c)^2$$

Now, $n_a-p_a\overset{?}{\le}\frac{s|b-c|}{2s+a}\Leftrightarrow n_a^2\overset{?}{\le}p_a^2+s^2.\frac{(b-c)^2}{(2s+a)^2}+2p_a.\frac{s|b-c|}{2s+a}$

$$\overset{\text{via }(\bullet\bullet\bullet\bullet)\text{ and }(\bullet\bullet\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s}{a}.(b-c)^2-s(s-a)-\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\overset{?}{\le}s^2.\frac{(b-c)^2}{(2s+a)^2}+2p_a.\frac{s|b-c|}{2s+a}$$

$$\Leftrightarrow\left(\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2}-\frac{s^2}{(2s+a)^2}\right)(b-c)^2\overset{?}{\le}2p_a.\frac{s|b-c|}{2s+a}\Leftrightarrow$$

$$\frac{4s^3|b-c|^2}{a(2s+a)}\le 2s.p_a.|b-c|\Leftrightarrow ap_a\overset{?}{\ge}\frac{2s^2}{2s+a}.|b-c|\ (\because |b-c|\ge 0)$$

$$\overset{\text{via }(\bullet\bullet\bullet\bullet)}{\Leftrightarrow} a^2\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\overset{?}{\ge}\frac{4s^4}{(2s+a)^2}.(b-c)^2$$

$$\Leftrightarrow a^2s(s-a)-\frac{4s^4-a^2s(3s+a)}{(2s+a)^2}.(b-c)^2\overset{?}{\ge}0$$

$$\Leftrightarrow a^2s(s-a)-\frac{s(s-a)(2s+a)^2}{(2s+a)^2}.(b-c)^2\overset{?}{\ge}0$$

$$\Leftrightarrow s(s-a)(a^2-(b-c)^2) \overset{?}{\geq} 0 \to \text{true} \therefore n_a - p_a \leq \frac{s|b-c|}{2s+a}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$= \sum_{\text{cyc}} a^3 - abc + 2a.2bc\cos A = 2s(s^2-6Rr-3r^2) - 4Rrs + 16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet\bullet\bullet\bullet\bullet\bullet)}{=} 2s(s^2-8Rr-3r^2+8Rr\cos A) \therefore (\bullet),(\bullet\bullet\bullet\bullet\bullet\bullet\bullet)$$

$$\Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}.2s(s^2-8Rr-3r^2+8Rr\cos A)$$

$$= \frac{4s^2}{(2s+a)^2}.\left(s^2-8Rr-3r^2+8Rr\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$= \frac{4s^2}{(2s+a)^2}.\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right) < \frac{4s^4}{(2s+a)^2} \Rightarrow p_a < \frac{2s^2}{2s+a} \therefore n_a - p_a$$

$$= \frac{n_a^2-p_a^2}{n_a+p_a} \geq \frac{\left(\frac{s}{a}-\frac{s(3s+a)}{(2s+a)^2}\right)(b-c)^2}{s+\frac{2s^2}{2s+a}} \quad (\because n_a^2 = s^2-2h_ar_a < s^2 \Rightarrow n_a < s)$$

$$= \frac{s(4s^2+sa)(b-c)^2}{a(2s+a)^2.\left(\frac{4s^2+sa}{2s+a}\right)} = \frac{s(b-c)^2}{a(2s+a)} \therefore n_a - p_a \geq \frac{s(b-c)^2}{a(2s+a)} \text{ and so,}$$

$$\frac{s(b-c)^2}{a(2s+a)} \leq n_a - p_a \leq \frac{s|b-c|}{2s+a} \ \forall \Delta \text{ ABC},$$

with equality iff Δ ABC is equilateral (QED)