ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with $p_a \to$ Spieker cevian, the following relationship holds :

$$\frac{2s(b-c)^2}{4s^2-a^2} \le p_a - w_a \le \frac{2sa|b-c|}{4s^2-a^2}$$

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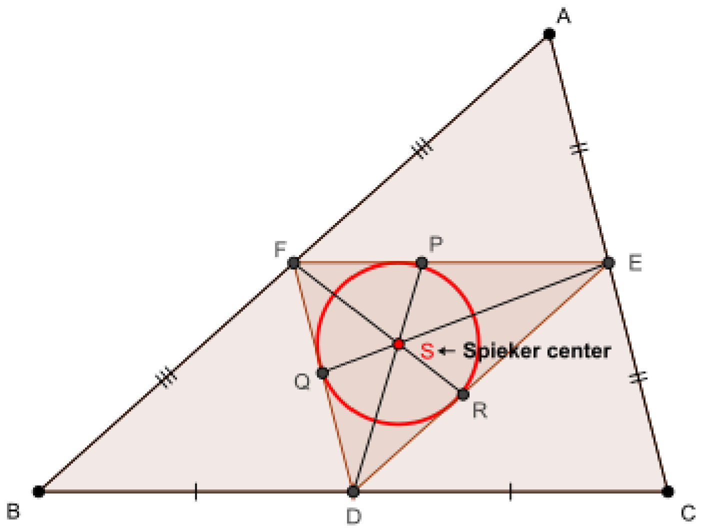

Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \to (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B+C}{2} = \frac{B+\pi-A}{2}$$

$$= \frac{\pi}{2} - \frac{A-B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A-C}{2} \to (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$= \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now, }\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a(b^2+c^2)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

$$\text{Again, }\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)(b^2+c^2+ab+ca)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

Now, $b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$

$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$

$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a - w_a \ge \frac{2s(b-c)^2}{4s^2-a^2} \Leftrightarrow p_a^2 - w_a^2 \ge \frac{4s^2(b-c)^4}{(4s^2-a^2)^2} + \frac{4s.w_a.(b-c)^2}{4s^2-a^2} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\frac{s(3s+a)}{(2s+a)^2} + \frac{s(s-a)}{(2s-a)^2} - \frac{4s^2(b-c)^2}{(4s^2-a^2)^2} \overset{(\blacksquare)}{\ge} \frac{4s.w_a}{4s^2-a^2} \quad (\because (b-c)^2 \ge 0)$$

$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2} + \frac{s(s-a)}{(2s-a)^2} - \frac{4s^2(b-c)^2}{(4s^2-a^2)^2} > \frac{s(3s+a)}{(2s+a)^2} + \frac{s(s-a)}{(2s-a)^2} - \frac{4s^2a^2}{(4s^2-a^2)^2}$$

$$= \frac{s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2 - 4s^2a^2}{(4s^2-a^2)^2} = \frac{8s^2(2s+a)(s-a)}{(4s^2-a^2)^2} > 0$$

$$\therefore (\blacksquare) \Leftrightarrow \frac{\left(s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2 - 4s^2(b-c)^2\right)^2}{(4s^2-a^2)^4} \ge \frac{16s^2}{(4s^2-a^2)^2}.\left(s(s-a) - \frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{16s^4(b-c)^4 - 8s^2(b-c)^2\left(s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2\right) + \left(s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2\right)^2}{(4s^2-a^2)^2} \ge \frac{16s^2\left(s(s-a)(2s-a)^2 - s(s-a)(b-c)^2\right)}{(2s-a)^2}$$

$$\Leftrightarrow 16s^4(b-c)^4 - 8s^2(b-c)^2\left(\begin{matrix} s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2 \\ -2s(s-a)(2s+a)^2 \end{matrix}\right)$$
$$+\left(s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2\right)^2 - 16s^3(s-a)(4s^2-a^2)^2 \ge 0$$
$$\Leftrightarrow 16s^4(b-c)^4 - 16s^3(b-c)^2(4s^3 - 4s^2a + sa^2 + a^3)$$
$$+16a^2s^3(4s^3 - 4s^2a + a^3) \ge 0$$

$$\Leftrightarrow s(b-c)^4 - (4s^3 - 4s^2a + sa^2 + a^3)(b-c)^2 + a^2(4s^3 - 4s^2a + a^3) \overset{(\blacksquare\blacksquare)}{\ge} 0$$

and in order to prove $(\blacksquare\blacksquare)$, it suffices to prove :

$$(b-c)^2 \le \frac{(4s^3 - 4s^2a + sa^2 + a^3) - \sqrt{\delta}}{2s}, \text{ where } \delta =$$
$$(4s^3 - 4s^2a + sa^2 + a^3)^2 - 4sa^2(4s^3 - 4s^2a + a^3) \text{ and } \because (b-c)^2 < a^2$$
$$\therefore \text{ it suffices to prove : } 2sa^2 \le (4s^3 - 4s^2a + sa^2 + a^3)$$
$$-\sqrt{(4s^3 - 4s^2a + sa^2 + a^3)^2 - 4sa^2(4s^3 - 4s^2a + a^3)}$$
$$\Leftrightarrow \sqrt{(s-a)^2(4s^2-a^2)^2} \le 4s^3 - 4s^2a - sa^2 + a^3 = (s-a)(4s^2-a^2) \to \text{true}$$

$$\Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true } \therefore p_a - w_a \ge \frac{2s(b-c)^2}{4s^2-a^2}$$

$$\text{Again, } p_a - w_a \le \frac{2sa|b-c|}{4s^2-a^2} \Leftrightarrow \frac{s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2}{(4s^2-a^2)^2}.(b-c)^2 \le \frac{4s^2a^2(b-c)^2}{(4s^2-a^2)^2} + \frac{4sa.w_a.|b-c|}{4s^2-a^2}$$

$$\Leftrightarrow \frac{s(3s+a)(2s-a)^2 + s(s-a)(2s+a)^2 - 4s^2a^2}{(4s^2-a^2)^2}.|b-c| \le \frac{4sa.w_a}{4s^2-a^2}$$

$$(\because |b-c| \geq 0) \Leftrightarrow \frac{8s^2(2s+a)(s-a)}{4s^2-a^2}.|b-c| \leq 4sa.w_a$$

$$\Leftrightarrow \frac{4s^2(2s+a)^2(s-a)^2}{(4s^2-a^2)^2}.(b-c)^2 \leq a^2\left(s(s-a) - \frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow \frac{4s^2(s-a)^2 + a^2s(s-a)}{(2s-a)^2}.(b-c)^2 \leq a^2s(s-a)$$

$$\Leftrightarrow \frac{s(s-a)(4s^2-4sa+a^2)}{(2s-a)^2}.(b-c)^2 \leq a^2s(s-a)$$

$$\Leftrightarrow s(s-a).(b-c)^2 \leq a^2s(s-a) \Leftrightarrow s(s-a)(a^2-(b-c)^2) \overset{?}{\geq} 0 \to \text{true}$$

$$\therefore p_a - w_a \leq \frac{2sa|b-c|}{4s^2-a^2} \text{ and so, } \frac{2s(b-c)^2}{4s^2-a^2} \leq p_a - w_a \leq \frac{2sa|b-c|}{4s^2-a^2}$$

$\forall \Delta$ ABC, with equality iff Δ ABC is equilateral (QED)