ROMANIAN MATHEMATICAL MAGAZINE

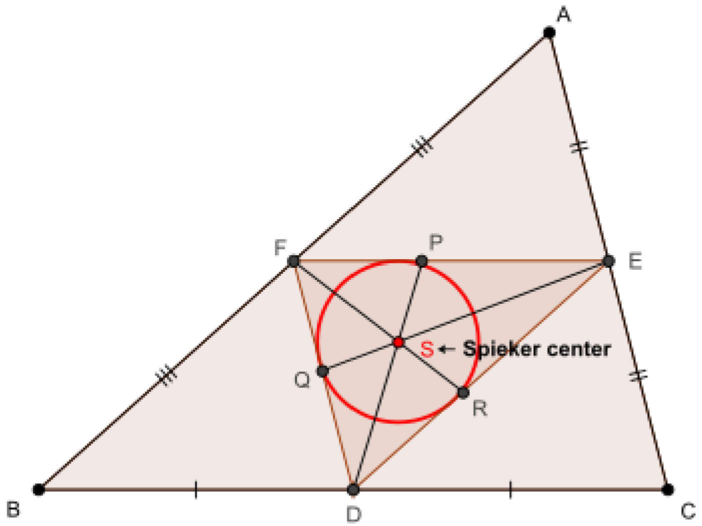
In any Δ ABC with p_a, p_b, p_c

$\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}+\frac{p_b+w_b}{2m_b+\sqrt{p_bw_b}}+\frac{p_c+w_c}{2m_c+\sqrt{p_cw_c}}\le 2$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $=r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,} \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\, AFS, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\, AES,\ b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs$$

$$\overset{\text{via (***) and (****)}}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2 \overset{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2 \overset{(\bullet)}{=} \frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2) = b^3+c^3-abc+a(2b^2+2c^2-a^2)$$
$$= (b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$
$$= 2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$
$$= (2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a).\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$
$$-\frac{a(b-c)^2}{4}\ (a=y+z, b=z+x, c=x+y)$$
$$= (2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$
$$= (2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore \boxed{b^3+c^3-abc+a(4m_a^2) \overset{(\bullet\bullet)}{=} (2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore (\bullet),(\bullet\bullet) \Rightarrow p_a^2 = \frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$
$$= s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$
$$= s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$= s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, }\frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}\le\frac{2}{3}\Leftrightarrow 3(p_a+w_a)-2\sqrt{p_aw_a}\le 4m_a$$

$$\Leftrightarrow 9(p_a+w_a)^2+4p_aw_a-12(p_a+w_a).\sqrt{p_aw_a}\le 16m_a^2 \text{ and } \because$$

$$-12(p_a+w_a).\sqrt{p_aw_a}\overset{A-G}{\le}-24p_aw_a \therefore \text{ it suffices to prove :}$$

$$9(p_a^2+w_a^2)+18p_aw_a+4p_aw_a-24p_aw_a\le 16m_a^2 \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$9\left(2s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)-16\left(s(s-a)+\frac{(b-c)^2}{4}\right)$$

$$\le 2p_aw_a \Leftrightarrow \left(\frac{9s(3s+a)}{(2s+a)^2}-\frac{9s(s-a)}{(2s-a)^2}-4\right).(b-c)^2\le 2p_aw_a-2s(s-a)$$

$$\Leftrightarrow \frac{4s^4-36s^3a+25s^2a^2+9sa^3-2a^4}{(4s^2-a^2)^2}.(b-c)^2 \boxed{\overset{(\blacksquare)}{\le}} p_aw_a-s(s-a)$$

$\boxed{\text{Case 1}}$ $4s^4-36s^3a+25s^2a^2+9sa^3-2a^4\le 0$ *and then* : LHS of $(\blacksquare)\le 0$

$$\le \text{RHS of }(\blacksquare)\left(\begin{array}{c}\because p_a^2=s(s-a)+\dfrac{s(3s+a)(b-c)^2}{(2s+a)^2}\ge m_a^2=s(s-a)+\dfrac{(b-c)^2}{4}\\ \Rightarrow p_aw_a-s(s-a)\ge m_aw_a-s(s-a)\overset{\text{Lascu}+A-G}{\ge}0\end{array}\right)$$

$\boxed{\text{Case 2}}$ $4s^4-36s^3a+25s^2a^2+9sa^3-2a^4>0$

$$\left(\Leftrightarrow t=\frac{s}{a}\ge 8.20584\ (\textit{approximately})\right)\text{ and then : }(\blacksquare)\overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{4s^4-36s^3a+25s^2a^2+9sa^3-2a^4}{(4s^2-a^2)^2}.(b-c)^2\right)^2\le$$

$$\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)-\frac{s(s-a)(b-c)^2}{(2s-a)^2}\right)$$

$$\Leftrightarrow\left(\frac{(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)^2}{(4s^2-a^2)^4}+\frac{s^2(3s+a)(s-a)}{(4s^2-a^2)^2}\right)(b-c)^4\le$$

$$\left(\begin{array}{c}\dfrac{s^2(3s+a)(s-a)}{(2s+a)^2}-\dfrac{s^2(s-a)^2}{(2s-a)^2}\\ -\dfrac{2s(s-a)(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)}{(4s^2-a^2)^2}\end{array}\right)(b-c)^2\Leftrightarrow$$

$$\frac{(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)^2+s^2(3s+a)(s-a)(4s^2-a^2)^2}{(4s^2-a^2)^4}.(b-c)^4$$

$$\le\frac{\left(\begin{array}{c}s^2(3s+a)(s-a)(2s-a)^2-s^2(s-a)^2(2s+a)^2\\ -2s(s-a)(4s^4-36s^3a+25s^2a^2+9sa^3-2a^4)\end{array}\right)}{(4s^2-a^2)^2}.(b-c)^2$$

$$\Leftrightarrow (b-c)^2\left(\frac{4(s-a)\begin{pmatrix}16s^7-64s^6a+300s^5a^2-128s^4a^3\\-135s^3a^4+13s^2a^5+8sa^6-a^7\end{pmatrix}}{(4s^2-a^2)^4}.(b-c)^2 - \frac{4sa(s-a)(16s^3-12s^2a-4sa^2+a^3)}{(4s^2-a^2)^2}\right)\le 0$$

$$\Leftrightarrow \begin{pmatrix}16s^7-64s^6a+300s^5a^2-128s^4a^3-135s^3a^4\\+13s^2a^5+8sa^6-a^7\end{pmatrix}(b-c)^2 \overset{(\blacksquare\blacksquare)}{\le}$$

$$sa(16s^3-12s^2a-4sa^2+a^3)(4s^2-a^2)^2 \; (\because (b-c)^2\ge 0 \text{ and } (s-a)>0)$$

Now, $16s^7-64s^6a+300s^5a^2-128s^4a^3-135s^3a^4+13s^2a^5+8sa^6-a^7$

$$=(s-a)(16s^6-48s^5a+252s^4a^2+124s^3a^3-11s^2a^4+2sa^5+10a^6)+9a^7$$

$$=(s-a)\begin{pmatrix}(16s^6+225s^4a^2-120s^5a)+27s^4a^2+113s^3a^3\\+11s^2a^3(s-a)+2sa^5+10a^6\end{pmatrix}+9a^7$$

$$=(s-a)\begin{pmatrix}(4s^3-15s^2a)^2+72s^5a+27s^4a^2+113s^3a^3\\+11s^2a^3(s-a)+2sa^5+10a^6\end{pmatrix}+9a^7\overset{s>a}{>}0$$

$$\text{and} \because (b-c)^2<a^2 \therefore \text{LHS of } (\blacksquare\blacksquare) <$$

$$\begin{pmatrix}16s^7-64s^6a+300s^5a^2-128s^4a^3-135s^3a^4\\+13s^2a^5+8sa^6-a^7\end{pmatrix}a^2\overset{?}{<}$$

$$sa(16s^3-12s^2a-4sa^2+a^3)(4s^2-a^2)^2\Leftrightarrow$$

$$256t^8-208t^7-128t^6-188t^5+176t^4+115t^3-17t^2-7t+1\overset{?}{>}0$$

$$\Leftrightarrow (t-1)\left((t-2)\begin{pmatrix}256t^6+560t^5+1040t^4+1812t^3\\+3532t^2+7087t+14180\end{pmatrix}+28359\right)\overset{?}{>}0$$

$\rightarrow$ true $\because t\ge 8.20584$ (approximately) $>2 \Rightarrow (t-1),(t-2)>0\Rightarrow(\blacksquare\blacksquare)$

$\Rightarrow(\blacksquare)$ is true (strict inequality) and combining both cases, $(\blacksquare)$ is true $\forall\,\Delta$ ABC

$$\Rightarrow\frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}\le\frac{2}{3} \text{ and analogs } \forall\,\Delta \text{ ABC}\Rightarrow$$

$$\frac{p_a+w_a}{2m_a+\sqrt{p_aw_a}}+\frac{p_b+w_b}{2m_b+\sqrt{p_bw_b}}+\frac{p_c+w_c}{2m_c+\sqrt{p_cw_c}}\le 2\;\forall\,\Delta \text{ ABC},$$

with equality iff Δ ABC is equilateral (QED)