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If p_a, p_b, p_c are the Spieker's cevians in ΔABC then:

$$\frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}\ge\frac{2}{R}$$

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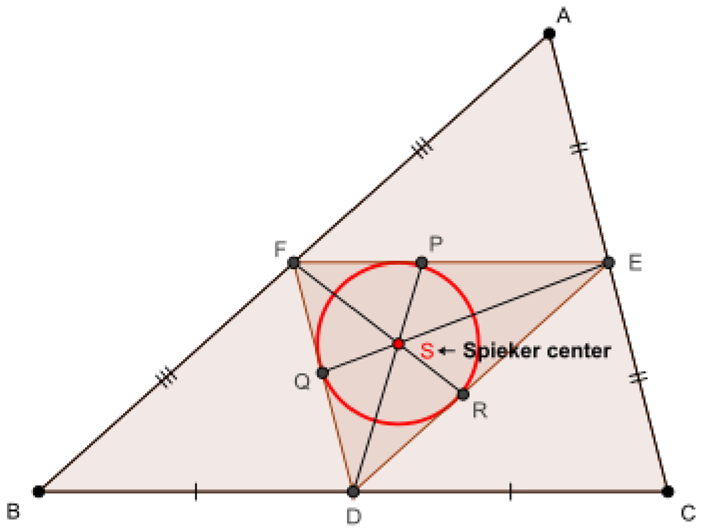

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\to(1)$$

$$\because \text{Spieker center is incenter of } \Delta \text{ DEF}, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\to(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$
$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$
$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$
$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$
$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a\left(b^2+c^2\right) - (b+c)(b-c)^2\right)$$
$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1 - 2\sin^2\frac{A}{2}\right)\right)}{2s}$$
$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$
$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$
$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$
$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$
$$= \frac{(a+b+c)\left(b^2+c^2+ab+ca\right) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$
$$= \frac{b^3+c^3-abc+a\left(2b^2+2c^2-a^2\right)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a\left(4m_a^2\right)}{4s}$$

$$\text{Via sine law on }\Delta\text{ AFS, }\frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$
$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\text{ AES, }b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, } [BAX]+[BAX]=[ABC]\Rightarrow \frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via } (***) \text{ and } (****)}{\Rightarrow} \frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, } b^3+c^3-abc+a(4m_a^2)=b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$=\sum_{\text{cyc}}a^3-abc+2a.2bc\cos A=2s(s^2-6Rr-3r^2)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2)\overset{(\blacksquare\blacksquare)}{=}2s(s^2-8Rr-3r^2+8Rr\cos A)\ \therefore (\blacksquare),(\blacksquare\blacksquare)$$

$$\Rightarrow p_a=\frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}$$

$$\Rightarrow \frac{1}{p_a}=\frac{1}{2s}.\frac{2s+a}{\sqrt{s^2-8Rr-3r^2+8Rr\cos A}}\ \text{and analogs}\Rightarrow \frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}$$

$$=\frac{1}{2s}.\sum_{\text{cyc}}\frac{(2s+a)^2}{\sqrt{s^2-8Rr-3r^2+8Rr\cos A}.(2s+a)}$$

$$\overset{\text{Bergstrom}}{\geq}\frac{1}{2s}.\frac{(6s+2s)^2}{\sum_{\text{cyc}}\left(\sqrt{s^2-8Rr-3r^2+8Rr\cos A}.(2s+a)\right)}$$

$$=\frac{1}{2s}.\frac{64s^2}{\sum_{\text{cyc}}\left(\sqrt{(s^2-8Rr-3r^2+8Rr\cos A)(2s+a)}.\sqrt{2s+a}\right)}$$

$$\overset{\text{CBS}}{\geq}\frac{1}{2s}.\frac{64s^2}{\sqrt{\sum_{\text{cyc}}((s^2-8Rr-3r^2+8Rr\cos A)(2s+a))}.\sqrt{6s+2s}}$$

$$=\frac{1}{2s}.\frac{64s^2}{\sqrt{(s^2-8Rr-3r^2)(8s)+16Rrs\left(\frac{R+r}{R}\right)+8Rr.\frac{2rs}{R}}.\sqrt{8s}}$$

$$\left(\because \sum_{\text{cyc}}a\cos A=\frac{2rs}{R}\right)=\frac{1}{2s}.\frac{64s^2}{8s.\sqrt{s^2-6Rr+r^2}}\overset{\text{Gerretsen}}{\geq}$$

$$\frac{4}{\sqrt{4R^2+4Rr+3r^2-6Rr+r^2}}=\frac{4}{\sqrt{4R^2-2Rr+4r^2}}\overset{\text{Euler}}{\geq}\frac{4}{\sqrt{4R^2-2Rr+2Rr}}=\frac{2}{R}$$

$$\therefore \frac{1}{p_a}+\frac{1}{p_b}+\frac{1}{p_c}\geq\frac{2}{R}\ \forall\,\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$