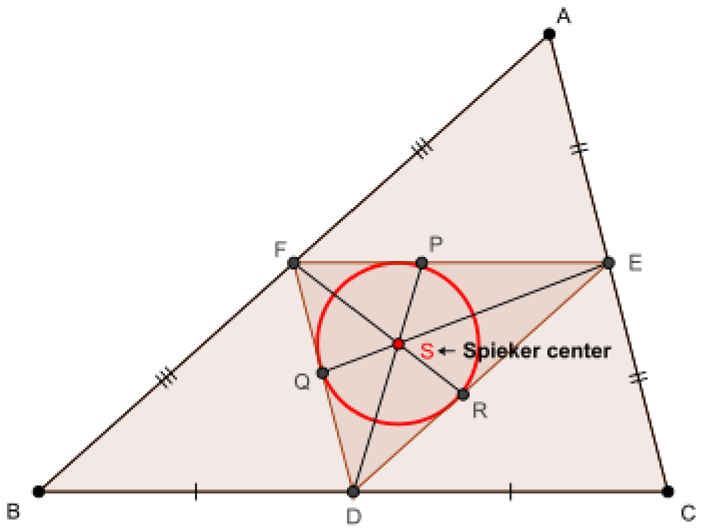
ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC with p_a, p_b, p_c
$\rightarrow$ Spieker cevians, the following relationship holds :

$$\frac{r_b + r_c}{r_a + p_a} + \frac{r_c + r_a}{r_b + p_b} + \frac{r_a + r_b}{r_c + p_c} \geq 3$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\measuredangle BAX) = \alpha$ and $m(\measuredangle CAX) = \beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right) - \sum\frac{a^4}{16} = \frac{1}{16}\left(2\sum a^2b^2 - \sum a^4\right) = \frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS) = B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2}$$

$$= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\measuredangle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2 = \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A - B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2\overset{(i)}{=}\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

Now, $\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}+\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$

$$=\frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2}+4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2}+2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$=Rr\left(1-2\sin^2\frac{B}{2}+1-2\sin^2\frac{C}{2}-2\left(1-2\sin^2\frac{A}{2}\right)\right)$$

$$=2Rr\left(\frac{2a(s-b)(s-c)-b(s-c)(s-a)-c(s-a)(s-b)}{abc}\right)$$

$$=\frac{Rr}{8Rrs}\left(2a^3+(b+c)a^2-2a\left(b^2+c^2\right)-(b+c)(b-c)^2\right)$$

$$=\frac{4(b+c)bc\sin^2\frac{A}{2}-2a.2bc\cos A}{8s}=\frac{bc\left((2s-a)\sin^2\frac{A}{2}-a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$=\frac{bc\left((2s+a)\sin^2\frac{A}{2}-a\right)}{2s}=\frac{(2s+a)(s-b)(s-c)}{2s}-2Rr$$

$$\Rightarrow-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=}\frac{-(2s+a)(s-b)(s-c)}{2s}+2Rr$$

Again, $\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}=\frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)}+\frac{ab}{(s-a)(s-b)}\right)$

$$=\frac{r^2}{4r^2s}\left(ca(s-b)+ab(s-c)\right)=\frac{ab+ca}{4}-2Rr\overset{(**)}{=}\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**)\Rightarrow 2AS^2=\frac{b^2+c^2+ab+ca}{4}-\frac{(2s+a)(s-b)(s-c)}{2s}$$

$$=\frac{(a+b+c)\left(b^2+c^2+ab+ca\right)-(2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$=\frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s}\Rightarrow 2AS^2\overset{(ii)}{=}\frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on }\Delta\,AFS,\ \frac{r}{2\sin\frac{C}{2}\sin\alpha}=\frac{AS}{\cos\frac{A-B}{2}}=\frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha\overset{(***)}{=}\frac{r(a+b)}{2AS}\text{ and via sine law on }\Delta\,AES,\ b\sin\beta\overset{(****)}{=}\frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}\cdot\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\ \boxed{p_a^2\overset{(\blacksquare)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3+a^3-abc+a(2b^2+2c^2-a^2)-a^3$$

$$=\sum_{cyc}a^3-abc+2a.2bc\cos A=2s(s^2-6Rr-3r^2)-4Rrs+16Rrs\cos A$$

$$\Rightarrow b^3+c^3-abc+a(4m_a^2)\overset{(\blacksquare\blacksquare)}{=}2s(s^2-8Rr-3r^2+8Rr\cos A)\ \therefore(\blacksquare),(\blacksquare\blacksquare)$$

$$\Rightarrow p_a=\frac{2s}{2s+a}.\sqrt{s^2-8Rr-3r^2+8Rr\cos A}\rightarrow(m)$$

$$\text{Also, }r_b+r_c=s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right)=\frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)}=4R\cos^2\frac{A}{2}$$

$$\therefore r_b+r_c\overset{(iii)}{=}4R\cos^2\frac{A}{2}$$

$$\text{We have}:\frac{r_b+r_c}{r_a+p_a}+\frac{r_c+r_a}{r_b+p_b}+\frac{r_a+r_b}{r_c+p_c}$$

$$=\sum_{cyc}\frac{(r_b+r_c)^2}{r_a(r_b+r_c)+p_a(r_b+r_c)}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}(r_b+r_c)\right)^2}{\sum_{cyc}r_a(r_b+r_c)+\sum_{cyc}p_a(r_b+r_c)}$$

$$=\frac{4(4R+r)^2}{2s^2+\sum_{cyc}p_a(r_b+r_c)}\overset{?}{\geq}3\Leftrightarrow\boxed{4(4R+r)^2-6s^2\underset{(\because)}{\overset{?}{\geq}}3\sum_{cyc}p_a(r_b+r_c)}$$

$$\text{We also have}:\prod_{cyc}(2s+a)=8s^3+4s^2\sum_{cyc}a+2s\sum_{cyc}ab+4Rrs$$

$$=8s^3+4s^2.2s+2s(s^2+4Rr+r^2)+4Rrs$$

$$\Rightarrow\prod_{cyc}(2s+a)\overset{(\blacksquare\blacksquare\blacksquare)}{=}2s(9s^2+6Rr+r^2)\text{ and}$$

$$\sum_{cyc}\frac{a(s-a)}{2s+a}=\frac{1}{\prod_{cyc}(2s+a)}.\sum_{cyc}a(s-a)(2s+b)(2s+c)\overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=}$$

$$\frac{1}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc} a(s-a)(8s^2-2sa+bc)$$

$$=\frac{1}{2s(9s^2+6Rr+r^2)}\cdot\left(\begin{array}{c}8s^2\left(s(2s)-2(s^2-4Rr-r^2)\right)\\ -2s\left(2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\right)+4Rrs\sum_{cyc}(s-a)\end{array}\right)$$

$$\Rightarrow \sum_{cyc}\frac{a(s-a)}{2s+a}\overset{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=}\frac{2rs(15R+2r)}{9s^2+6Rr+r^2}$$

Moreover, $\sum_{cyc}\frac{a^2}{2s+a}\overset{\text{via }(\blacksquare\blacksquare\blacksquare)}{=}\frac{1}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc} a^2(2s+b)(2s+c)$

$$=\frac{1}{2s(9s^2+6Rr+r^2)}\cdot\sum_{cyc} a^2(8s^2-2sa+bc)$$

$$=\frac{1}{2s(9s^2+6Rr+r^2)}\cdot\left(8s^2.2(s^2-4Rr-r^2)-4s^2(s^2-6Rr-3r^2)+4Rrs(2s)\right)$$

$$\Rightarrow \sum_{cyc}\frac{a^2}{2s+a}\overset{(\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare)}{=}\frac{2s(3s^2-8Rr-r^2)}{9s^2+6Rr+r^2}$$

Now, $\sum_{cyc} p_a(r_b+r_c)\overset{\text{via (m) and (iii)}}{=}$

$$\sum_{cyc}\left(\frac{2s.4R}{2s+a}\cdot\sqrt{s^2-8Rr-3r^2+8Rr\cos A}\cdot\frac{sa(s-a)}{abc}\right)$$

$$=\frac{8Rs^2}{4Rrs}\cdot\sum_{cyc}\left(\sqrt{\frac{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)a(s-a)}{2s+a}}\cdot\sqrt{\frac{a(s-a)}{2s+a}}\right)$$

$$\overset{CBS}{\leq}\frac{2s}{r}\cdot\sqrt{\sum_{cyc}\frac{\left(s^2-3r^2-16Rr\sin^2\frac{A}{2}\right)a(s-a)}{2s+a}}\cdot\sqrt{\sum_{cyc}\frac{a(s-a)}{2s+a}}$$

$$=\frac{2s}{r}\cdot\sqrt{(s^2-3r^2).\sum_{cyc}\frac{a(s-a)}{2s+a}-\frac{16Rr(s-b)(s-c)(s-a)}{abc}\cdot\sum_{cyc}\frac{a^2}{2s+a}}\cdot\sqrt{\sum_{cyc}\frac{a(s-a)}{2s+a}}$$

$$\overset{\text{via }(\blacksquare\blacksquare\blacksquare\blacksquare)\text{ and }(\blacksquare\blacksquare\blacksquare\blacksquare\blacksquare)}{=}$$

$$\frac{2s}{r}\cdot\sqrt{(s^2-3r^2).\frac{2rs(15R+2r)}{9s^2+6Rr+r^2}-\frac{16Rr.r^2s}{4Rrs}\cdot\frac{2s(3s^2-8Rr-r^2)}{9s^2+6Rr+r^2}}\cdot\sqrt{\frac{2rs(15R+2r)}{9s^2+6Rr+r^2}}$$

$$=\frac{4s^2}{9s^2+6Rr+r^2}\cdot\sqrt{(225R^2-120Rr-20r^2)s^2-r^2(195R^2+56Rr+4r^2)}$$

$$\Rightarrow \boxed{3\sum_{cyc} p_a(r_b+r_c)\leq\frac{12s^2}{9s^2+6Rr+r^2}\cdot\sqrt{\begin{array}{c}(225R^2-120Rr-20r^2)s^2\\ -r^2(195R^2+56Rr+4r^2)\end{array}}}$$

$$\overset{?}{\leq} 4(4R+r)^2-6s^2 \Leftrightarrow \left(2(4R+r)^2-3s^2\right)^2\left(9s^2+6Rr+r^2\right)^2$$

$$\overset{?}{\geq} 36s^4\left(\left(225R^2-120Rr-20r^2\right)s^2-r^2\left(195R^2+56Rr+4r^2\right)\right)$$

$$\Leftrightarrow 729s^8-\left(23652R^2+2484Rr+90r^2\right)s^6$$

$$+\left(82944R^4+62208R^3r+24624R^2r^2+4284Rr^3+261r^4\right)s^4$$

$$+r\left(110592R^5+122112R^4r+54144R^3r^2+12048R^2r^3+1344Rr^4+60r^5\right)s^2$$

$$+r^2\begin{pmatrix}36864R^6+49152R^5r+27136R^4r^2+7936R^3r^3\\+1296R^2r^4+112Rr^5+4r^6\end{pmatrix}\boxed{\overset{?}{\underset{(\bullet)}{\geq}}}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ and $s^2-(m+n)\leq 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0$$

$$\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0 \Rightarrow s^4-s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3\overset{(\blacklozenge)}{\leq}0$$

$\therefore 729\left(s^4-s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3\right)^2\geq 0 \therefore$ in order to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet)\geq$

$$729\left(s^4-s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3\right)^2$$

$$\Leftrightarrow -\left(17820R^2-26676Rr+3006r^2\right)s^6$$

$$+\left(71280R^4-147744R^3r-325296R^2r^2+45108Rr^3-4113r^4\right)s^4$$

$$+r\begin{pmatrix}483840R^5+2268288R^4r+1337184R^3r^2\\+227832R^2r^3-4488Rr^4-2856r^5\end{pmatrix}s^2$$

$$-r^2\begin{pmatrix}2949120R^6+4429824R^5r+2772224R^4r^2+925184R^3r^3\\+173664R^2r^4+17384Rr^5+725r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet)}{\geq}}0 \textit{ and}$$

$$\because -\left(17820R^2-26676Rr+3006r^2\right)s^2\begin{pmatrix}s^4-s^2\left(4R^2+20Rr-2r^2\right)\\+r(4R+r)^3\end{pmatrix}\overset{via\,(\blacklozenge)}{\geq}0$$

$\therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : LHS of $(\bullet\bullet)\geq$

$$-\left(17820R^2-26676Rr+3006r^2\right)s^2\left(s^4-s^2\left(4R^2+20Rr-2r^2\right)+r(4R+r)^3\right)$$

$$\Leftrightarrow -\left(397440R^3-231840R^2r+68364Rr^2-1899r^3\right)s^4$$

$$+\begin{pmatrix}1624320R^5+1416384R^4r+462960R^3r^2\\+69828R^2r^3+4908Rr^4+150r^5\end{pmatrix}s^2$$

$$-r\begin{pmatrix}2949120R^6+4429824R^5r+2772224R^4r^2+925184R^3r^3\\+173664R^2r^4+17384Rr^5+725r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet\bullet)}{\geq}}0 \textit{ and } \because$$

$$-\left(397440R^3-231840R^2r+68364Rr^2-1899r^3\right)\begin{pmatrix}s^4-s^2\left(4R^2+20Rr-2r^2\right)\\+r(4R+r)^3\end{pmatrix}$$

$\overset{via\,(\blacklozenge)}{\geq}0 \therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : LHS of $(\bullet\bullet\bullet)\geq$

$$-\left(397440R^3-231840R^2r+68364Rr^2-1899r^3\right)\begin{pmatrix}s^4-s^2\left(4R^2+20Rr-2r^2\right)\\+r(4R+r)^3\end{pmatrix}$$

$$\Leftrightarrow \left(1080R^5-175158R^4r+175662R^3r^2-54798R^2r^3+5613Rr^4-114r^5\right)s^2$$

$$+r\begin{pmatrix}702720R^6-5952R^5r-148624R^4r^2-4684R^3r^3\\+10116R^2r^4+881Rr^5-82r^6\end{pmatrix}\boxed{\overset{(\bullet\bullet\bullet\bullet)}{\geq}}0$$

$\boxed{Case\ 1}$ $1080R^5 - 175158R^4r + 175662R^3r^2 - 54798R^2r^3 + 5613Rr^4 - 114r^5 \geq 0$ *and then* : LHS of $(\bullet\bullet\bullet\bullet) \geq$

$$r\begin{pmatrix}702720R^6 - 5952R^5r - 148624R^4r^2 - 4684R^3r^3 + \\ 10116R^2r^4 + 881Rr^5 - 82r^6\end{pmatrix}$$

$$= r\left((R-2r)\begin{pmatrix}702720R^5 + 1399488R^4r + 2650352R^3r^2 \\ +5296020R^2r^3 + 10602156Rr^4 + 21205193r^5\end{pmatrix} + 42410304r^6\right)$$

$$\overset{\text{Euler}}{\geq} 42410304r^7 > 0 \Rightarrow (\bullet\bullet\bullet\bullet) \text{ is true (strict inequality)}$$

$\boxed{Case\ 2}$ $1080R^5 - 175158R^4r + 175662R^3r^2 - 54798R^2r^3 + 5613Rr^4 - 114r^5 < 0$ *and then* : LHS of $(\bullet\bullet\bullet\bullet)$

$$= -\left(-\begin{pmatrix}1080R^5 - 175158R^4r + 175662R^3r^2 - 54798R^2r^3 \\ +5613Rr^4 - 114r^5\end{pmatrix}\right)s^2$$

$$+r\begin{pmatrix}702720R^6 - 5952R^5r - 148624R^4r^2 - 4684R^3r^3 \\ +10116R^2r^4 + 881Rr^5 - 82r^6\end{pmatrix} \overset{\text{Gerretsen}}{\geq}$$

$$-\left(-\begin{pmatrix}1080R^5 - 175158R^4r + 175662R^3r^2 - 54798R^2r^3 \\ +5613Rr^4 - 114r^5\end{pmatrix}\right)(4R^2 + 4Rr + 3r^2)$$

$$+r\begin{pmatrix}702720R^6 - 5952R^5r - 148624R^4r^2 - 4684R^3r^3 \\ +10116R^2r^4 + 881Rr^5 - 82r^6\end{pmatrix} \overset{?}{\geq} 0$$

$$\Leftrightarrow 2160t^7 + 3204t^6 - 348t^5 - 95321t^4 + 162781t^3$$

$$-66141t^2 + 8632t - 212 \overset{?}{\geq} 0 \quad \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left((t-2)\begin{pmatrix}2160t^5 + 11844t^4 + 38388t^3 + 10855t^2 \\ +52649t + 101035\end{pmatrix} + 202176\right) \overset{?}{\geq} 0$$

$\rightarrow$ true $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet\bullet\bullet)$ is true $\therefore$ combining both *cases*, $(\bullet\bullet\bullet\bullet)$

$\Rightarrow (\bullet\bullet\bullet) \Rightarrow (\bullet\bullet) \Rightarrow (\bullet)$ is true $\forall\, \Delta$ ABC $\therefore \dfrac{r_b + r_c}{r_a + p_a} + \dfrac{r_c + r_a}{r_b + p_b} + \dfrac{r_a + r_b}{r_c + p_c} \geq 3$

$\forall\, \Delta$ ABC, " = " iff Δ ABC is equi*lateral* (QED)