ROMANIAN MATHEMATICAL MAGAZINE

In any non − obtuse Δ ABC, the following relationship holds :

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2 A+\sin^2 B+\sin^2 C)\ge 3$$

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$\boxed{\text{Case 1}}$ Δ ABC is right triangle and then : $s = 2R + r$ and

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2 A+\sin^2 B+\sin^2 C)\ge 3$$

$$\Leftrightarrow\left(\sum_{cyc}a^2\right).\frac{\left(\sum_{cyc}a^2\right)^2+\sum_{cyc}a^2b^2}{\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^2b^2\right)-a^2b^2c^2}+\frac{2}{3.4R^2}.\left(\sum_{cyc}a^2\right)\ge 6$$

$$\Leftrightarrow\frac{(s^2-4Rr-r^2)\left(4(s^2-4Rr-r^2)^2+(s^2+4Rr+r^2)^2-16R^2r^2s^2\right)}{(s^2-4Rr-r^2)((s^2+4Rr+r^2)^2-16R^2r^2s^2)-8R^2r^2s^2}$$
$$+\frac{s^2-4Rr-r^2}{3R^2}\ge 6$$

$$\Leftrightarrow s^8-(3R^2+16Rr)s^6+r(36R^3+37R^2r+16Rr^2-2r^3)s^4$$
$$-r^3(56R^3+69R^2r+16Rr^2)s^2$$
$$+r^3(192R^5+400R^4r+292R^3r^2+99R^2r^3+16Rr^4+r^5)\overset{(1)}{\ge}0$$

and putiing $s = 2R + r$ in LHS of (1), we have $(1)\Leftrightarrow R^4-4R^2r^2-4Rr^3-r^4\ge 0$

$$\Leftrightarrow(t^2-2t-1)(t+1)^2\ge 0\ \left(t=\frac{R}{r}\right)\Leftrightarrow t^2-2t-1\overset{(2)}{\ge}0$$

WLOG we may assume $A = 90°$ and then : $a^2=b^2+c^2\overset{A-G}{\ge}\frac{2bca}{a}$

$$\Rightarrow 8R^3\sin^3 90°=8Rrs\Rightarrow R^2\ge r(2R+r)\Rightarrow t^2-2t-1\ge 0\Rightarrow(2)\Rightarrow(1)\text{ is true}$$

$$\therefore\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2 A+\sin^2 B+\sin^2 C)\ge 3$$

is true ∀ right Δ ABC

$\boxed{\text{Case 2}}$ Δ ABC is acute triangle and $\because b^2+c^2>a^2$ and analogs

$\therefore a^2, b^2, c^2$ form sides of a triangle XYZ (say)

$$\Rightarrow\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}(\sin^2 A+\sin^2 B+\sin^2 C)\ge 3$$

$$\Leftrightarrow\sum_{cyc}\frac{a^2}{b^2+c^2}+\frac{2}{3}.\frac{1}{4.\frac{a^2b^2c^2}{2\sum_{cyc}a^2b^2-\sum_{cyc}a^4}}.\sum_{cyc}a^2$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{y+z}+\frac{2}{3}.\frac{1}{4.\frac{xyz}{2\sum_{yc}xy-\sum_{cyc}x^2}}.\sum_{cyc}x\ge 3$$

$$\Leftrightarrow \sum_{cyc}\frac{x}{y+z}+\frac{1}{6}\cdot\frac{2\sum_{yc}xy-\sum_{cyc}x^2}{xyz}\cdot\sum_{cyc}x \overset{(*)}{\geq} 3$$

Now, we shall prove that $\forall\,\Delta\,ABC:\ \sum_{cyc}\frac{a}{b+c}+\frac{1}{6}\cdot\frac{2\sum_{yc}ab-\sum_{cyc}a^2}{abc}\cdot\sum_{cyc}a \overset{(\bullet)}{\geq} 3$

$$(\bullet) \Leftrightarrow \left(\sum_{cyc}a\right)\cdot\frac{\sum_{cyc}(c+a)(a+b)}{2s(s^2+2Rr+r^2)}+\frac{1}{6}\cdot\frac{2(s^2+4Rr+r^2)-2(s^2-4Rr-r^2)}{4Rrs}\cdot 2s$$

$$\geq 6 \Leftrightarrow \frac{2s\left(\left(\sum_{cyc}a^2+2\sum_{cyc}ab\right)+\sum_{cyc}ab\right)}{2s(s^2+2Rr+r^2)}+\frac{(16Rr+4r^2)(2s)}{24Rrs} \geq 6$$

$$\Leftrightarrow \frac{4s^2+s^2+4Rr+r^2}{s^2+2Rr+r^2}+\frac{4R+r}{3R} \geq 6 \Leftrightarrow (R+r)s^2 \overset{(\bullet\bullet)}{\geq} r(16R^2+9Rr-r^2)$$

We have : $(R+r)s^2 \overset{Gerretsen}{\geq} (R+r)(16Rr-5r^2) \overset{?}{\geq} r(16R^2+9Rr-r^2)$

$\Leftrightarrow 2Rr \overset{?}{\geq} 4r^2 \rightarrow$ true via Euler $\Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (*)$ is true

$$\therefore \frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}+\frac{2}{3}\left(\sin^2 A+\sin^2 B+\sin^2 C\right) \geq 3\ \forall\ acute\ \Delta\,ABC$$

$\therefore$ combining both cases, $\frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}$

$+\frac{2}{3}\left(\sin^2 A+\sin^2 B+\sin^2 C\right) \geq 3\ \forall$ non − obtuse $\Delta\,ABC$,

" = " iff $\Delta\,ABC$ is right or equilateral (QED)