ROMANIAN MATHEMATICAL MAGAZINE

In any non − obtuse Δ ABC with ω → Brocard's angle, the following relationship holds :

$$\csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \ge 13$$

When does equality hold ?

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$\boxed{\text{Case 1}}$ Δ ABC is right triangle and then : $s = 2R + r$ and

$$\csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \ge 13$$

$$\Leftrightarrow \frac{(s^2+4Rr+r^2)^2 - 16Rrs^2}{4r^2s^2} + \frac{s^2-4Rr-r^2}{R^2} \ge 13$$

$$\Leftrightarrow R^2\left(\left((2R+r)^2+4Rr+r^2\right)^2 - 16Rr(2R+r)^2\right)$$

$$+8r^2(2R+r)^2\left((2R+r)^2-4Rr-r^2\right) \ge 52R^2r^2(2R+r)^2$$

$$\Leftrightarrow R^4 - 4R^2r^2 - 4Rr^3 - r^4 \ge 0 \Leftrightarrow (t^2-2t-1)(t+1)^2 \ge 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow t^2 - 2t - 1 \overset{(1)}{\ge} 0$$

WLOG we may assume $A = 90°$ and then : $a^2 = b^2 + c^2 \overset{A-G}{\ge} \frac{2bca}{a} \Rightarrow 8R^3\sin^3 90°$

$= 8Rrs \Rightarrow R^2 \ge r(2R+r) \Rightarrow t^2 - 2t - 1 \ge 0 \Rightarrow (2) \Rightarrow (1)$ is true

$\therefore \csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \ge 13\ \forall$ right Δ ABC

$\boxed{\text{Case 2}}$ Δ ABC is acute triangle and $\because b^2 + c^2 > a^2$ and analogs

$\therefore a^2, b^2, c^2$ form sides of a triangle XYZ (say)

$$\Rightarrow \csc^2\omega + 4(\sin^2 A + \sin^2 B + \sin^2 C) \ge 13 \Leftrightarrow$$

$$\frac{4\sum_{cyc} a^2b^2}{2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4} + \frac{\sum_{cyc} a^2}{\frac{a^2b^2c^2}{2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4}} \ge 13$$

$$\Leftrightarrow \boxed{\frac{4\sum_{yc} xy}{2\sum_{yc} xy - \sum_{cyc} x^2} + \frac{\left(\sum_{cyc} x\right)\left(2\sum_{yc} xy - \sum_{cyc} x^2\right)}{xyz} \overset{(*)}{\ge} 13}$$

Now, we shall prove that ∀ Δ ABC :

$$\frac{4\sum_{yc} ab}{2\sum_{yc} ab - \sum_{cyc} a^2} + \frac{\left(\sum_{cyc} a\right)\left(2\sum_{yc} ab - \sum_{cyc} a^2\right)}{abc} \overset{(\bullet)}{\ge} 13$$

$$(\bullet) \Leftrightarrow \frac{4(s^2+4Rr+r^2)}{2(s^2+4Rr+r^2) - 2(s^2-4Rr-r^2)}$$

$$+\frac{2s\left(2(s^2+4Rr+r^2) - 2(s^2-4Rr-r^2)\right)}{4Rrs} \ge 13$$

$$\Leftrightarrow \frac{R(s^2+4Rr+r^2) + 2r(4R+r)^2}{Rr(4R+r)} \ge 13 \Leftrightarrow \boxed{Rs^2 \overset{(\bullet\bullet)}{\ge} r(16R^2 - 4Rr - 2r^2)}$$

$$\text{Now, via Rouche, } Rs^2 \geq R\left(2R^2 + 10Rr - r^2 - 2(R-2r).\sqrt{R^2 - 2Rr}\right)$$

$$\overset{?}{\geq} r\left(16R^2 - 4Rr - 2r^2\right) \Leftrightarrow 2R^3 - 6R^2r + 3Rr^2 + 2r^3 \overset{?}{\geq} 2R(R-2r).\sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow \boxed{(R-2r)\left(2R^2 - 2Rr - r^2\right) \underbrace{\overset{?}{\geq}}_{(\bullet\bullet)} 2R(R-2r).\sqrt{R^2 - 2Rr}} \text{ and } \because R - 2r \overset{\text{Euler}}{\geq} 0$$

$$\therefore \text{ in order to prove } (\bullet\bullet), \text{ it suffices to prove : } \left(2R^2 - 2Rr - r^2\right)^2 > 4R^2\left(R^2 - 2Rr\right)$$

$$\Leftrightarrow 4Rr^3 + r^4 > 0 \rightarrow \text{true} \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \Rightarrow (*) \text{ is true}$$

$$\therefore \csc^2\omega + 4\left(\sin^2 A + \sin^2 B + \sin^2 C\right) \geq 13 \ \forall \text{ acute } \Delta ABC$$

$$\therefore \text{ combining both cases, } \csc^2\omega + 4\left(\sin^2 A + \sin^2 B + \sin^2 C\right) \geq 13$$

$$\forall \text{ non} - \text{obtuse } \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is right or equilateral (QED)}$$