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If p_a, p_b, p_c
$\rightarrow$ Spieker cevians in Δ ABC, then the following relationship holds :

$$\frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c}\geq 3+\frac{2}{3}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{F}$$

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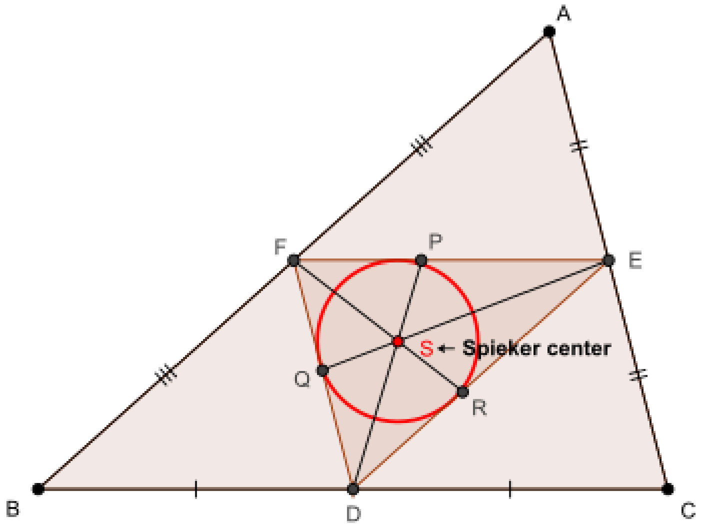

Let AS produced meet BC at X and $m(\measuredangle BAX)=\alpha$ and $m(\measuredangle CAX)=\beta$ (say) and inradius of Δ DEF $= r'$ (say)

$$\text{Now, } 16[DEF]^2=2\sum\left(\frac{a^2}{4}\right)\left(\frac{b^2}{4}\right)-\sum\frac{a^4}{16}=\frac{1}{16}\left(2\sum a^2b^2-\sum a^4\right)=\frac{16r^2s^2}{16}$$

$$\Rightarrow [DEF]=\frac{rs}{4}\Rightarrow r'\left(\frac{\frac{a}{2}+\frac{b}{2}+\frac{c}{2}}{2}\right)=\frac{rs}{4}\Rightarrow r'=\frac{r}{2}\rightarrow(1)$$

$$\because \text{ Spieker center is incenter of } \Delta\, DEF, \therefore m(\measuredangle AFS)=B+\frac{C}{2}=\frac{2B+C}{2}=\frac{B+\pi-A}{2}$$

$$=\frac{\pi}{2}-\frac{A-B}{2} \text{ and } m(\measuredangle AES)=C+\frac{B}{2}=\frac{\pi}{2}-\frac{A-C}{2}\rightarrow(2)$$

Via (1), (2) and using cosine law on Δ AFS and Δ AES, we arrive at :

$$AS^2=\frac{r^2}{4\sin^2\frac{C}{2}}+\frac{c^2}{4}-\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2}$$

$$=\frac{r^2}{4\sin^2\frac{B}{2}}+\frac{b^2}{4}-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\Rightarrow 2AS^2 \overset{(i)}{=} \frac{r^2}{4\sin^2\frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{b^2}{4}$$

$$-\left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\text{Now,}\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$= \frac{r}{2}\left(4R\cos\frac{C}{2}\sin\frac{A-B}{2} + 4R\cos\frac{B}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(2\sin\frac{A+B}{2}\sin\frac{A-B}{2} + 2\sin\frac{A+C}{2}\sin\frac{A-C}{2}\right)$$

$$= Rr\left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2\left(1 - 2\sin^2\frac{A}{2}\right)\right)$$

$$= 2Rr\left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc}\right)$$

$$= \frac{Rr}{8Rrs}\left(2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2\right)$$

$$= \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a.2bc\cos A}{8s} = \frac{bc\left((2s-a)\sin^2\frac{A}{2} - a\left(1-2\sin^2\frac{A}{2}\right)\right)}{2s}$$

$$= \frac{bc\left((2s+a)\sin^2\frac{A}{2} - a\right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr$$

$$\Rightarrow -\left(\frac{2r}{2\sin\frac{C}{2}}\right)\left(\frac{c}{2}\right)\sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}}\right)\left(\frac{b}{2}\right)\sin\frac{A-C}{2}$$

$$\overset{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr$$

$$\text{Again,}\frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4}\left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)}\right)$$

$$= \frac{r^2}{4r^2s}\left(ca(s-b) + ab(s-c)\right) = \frac{ab+ca}{4} - 2Rr \overset{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}}$$

$$(i),(*),(**) \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s}$$

$$= \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{8s}$$

$$= \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \overset{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s}$$

$$\text{Via sine law on } \Delta\text{ AFS}, \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{cAS}{(a+b)\sin\frac{C}{2}}$$

$$\Rightarrow c\sin\alpha \overset{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \Delta\text{ AES}, b\sin\beta \overset{(****)}{=} \frac{r(a+c)}{2AS}$$

$$\text{Now, }[BAX]+[BAX]=[ABC]\Rightarrow\frac{1}{2}p_a c\sin\alpha+\frac{1}{2}p_a b\sin\beta=rs$$

$$\overset{\text{via }(***)\text{ and }(****)}{\Rightarrow}\frac{p_a(a+b+a+c)}{4AS}=s\Rightarrow p_a=\frac{4s}{2s+a}AS$$

$$\Rightarrow p_a^2\overset{\text{via (ii)}}{=}\frac{16s^2}{(2s+a)^2}.\frac{b^3+c^3-abc+a(4m_a^2)}{8s}$$

$$\therefore\boxed{p_a^2\overset{(\bullet)}{=}\frac{2s}{(2s+a)^2}\left(b^3+c^3-abc+a(4m_a^2)\right)}$$

$$\text{Now, }b^3+c^3-abc+a(4m_a^2)=b^3+c^3-abc+a(2b^2+2c^2-a^2)$$

$$=(b+c)(b^2-bc+c^2)+a(b^2-bc+c^2)+a(b^2+c^2-a^2)$$

$$=2s(b^2-bc+c^2)+a(b^2-bc+c^2+bc-a^2)$$

$$=(2s+a)(b^2-bc+c^2)+a\left(\frac{(b+c)^2-(b-c)^2}{4}-a^2\right)$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(b+c+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)(b^2-bc+c^2)+\frac{a(2s-a+2a)(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).\frac{4b^2+4c^2-4bc+a(b+c-2a)}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a).$$

$$\frac{4(z+x)^2+4(x+y)^2-4(z+x)(x+y)+(y+z)\left((z+x)+(x+y)-2(y+z)\right)}{4}$$

$$-\frac{a(b-c)^2}{4}\ (a=y+z,b=z+x,c=x+y)$$

$$=(2s+a).\frac{4x(x+y+z)+2x(y+z)+3(y-z)^2}{4}-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{a(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{3}{4}(b-c)^2+\frac{a(s-a)}{2}\right)-\frac{(a+2s-2s)(b-c)^2}{4}$$

$$=(2s+a)\left(s(s-a)+\frac{(b-c)^2}{2}+\frac{a(s-a)}{2}\right)+\frac{s(b-c)^2}{2}$$

$$\therefore\boxed{b^3+c^3-abc+a(4m_a^2)\overset{(\bullet\bullet)}{=}(2s+a)\left(\frac{(s-a)(2s+a)}{2}+\frac{(b-c)^2}{2}\right)+\frac{s(b-c)^2}{2}}$$

$$\therefore(\bullet),(\bullet\bullet)\Rightarrow p_a^2=\frac{2s}{(2s+a)^2}\left(\frac{(s-a)(2s+a)^2}{2}+\frac{(2s+a)(b-c)^2}{2}+\frac{s(b-c)^2}{2}\right)$$

$$=s(s-a)+(b-c)^2\left(\left(\frac{s}{2s+a}\right)^2+\frac{s}{2s+a}+\frac{1}{4}-\frac{1}{4}\right)$$

$$=s(s-a)-\frac{(b-c)^2}{4}+(b-c)^2.\left(\frac{s}{2s+a}+\frac{1}{2}\right)^2$$

$$=s(s-a)+\frac{(b-c)^2}{4}\left(\frac{(4s+a)^2}{(2s+a)^2}-1\right)$$

$$\Rightarrow p_a^2 \overset{(\bullet\bullet\bullet)}{=} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\text{Now, } p_a \overset{?}{\geq} h_a+\frac{2}{3}\cdot\frac{(b-c)^2}{a} \overset{\text{via }(\bullet\bullet\bullet)}{\Leftrightarrow} s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}$$

$$\overset{?}{\geq} s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}+\frac{4}{9}\cdot\frac{(b-c)^4}{a^2}+\frac{4h_a}{3}\cdot\frac{(b-c)^2}{a}$$

$$\Leftrightarrow \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2} \boxed{\overset{?}{\geq}_{(\blacksquare)}} \frac{4h_a}{3a}\ (\because (b-c)^2 \geq 0)$$

$$\text{We have : } \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4(b-c)^2}{9a^2} \overset{\because a^2>(b-c)^2}{>} \frac{s(3s+a)}{(2s+a)^2}+\frac{s(s-a)}{a^2}-\frac{4}{9}$$

$$=\frac{9s(3s+a)a^2+9s(s-a)(2s+a)^2-4a^2(2s+a)^2}{9a^2(2s+a)^2}$$

$$=\frac{4(s-a)\left(9s^3+9s^2a+5sa^2+a^3\right)}{9a^2(2s+a)^2} \overset{s>a}{>} 0 \Rightarrow \text{LHS of } (\blacksquare)>0 \therefore (\blacksquare) \Leftrightarrow$$

$$\frac{T^2}{a^4(2s+a)^4}+\frac{16(b-c)^4}{81a^4}-\frac{8T(b-c)^2}{9a^4(2s+a)^2} \overset{?}{\geq} \frac{16}{9a^2}\cdot\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)$$

$$\left(T=s(3s+a)a^2+s(s-a)(2s+a)^2\right)$$

$$\Leftrightarrow \frac{16(b-c)^4}{81a^4}-\frac{8(b-c)^2}{9a^4}\left(\frac{T}{(2s+a)^2}-2s(s-a)\right)+\frac{T^2}{a^4(2s+a)^4}-\frac{16s(s-a)}{9a^2}$$

$$\Leftrightarrow \left(\frac{4(b-c)^2}{9}\right)^2+\frac{4(b-c)^2}{9}\cdot\frac{4s\left(2s^3-3sa^2-a^3\right)}{(2s+a)^2}$$

$$+\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4} \boxed{\overset{?}{\geq}_{(\blacksquare\blacksquare)}} 0$$

$$\text{Now, LHS of } (\blacksquare\blacksquare) \text{ is a quadratic polynomial in } ''\frac{4(b-c)^2}{9}'' \text{ whose } \boxed{\text{discriminant}}$$

$$=\frac{16s^2\left(2s^3-3sa^2-a^3\right)^2}{(2s+a)^4}-4\cdot\frac{9T^2-16s(s-a)a^2(2s+a)^4}{9(2s+a)^4}$$

$$=-\frac{16sa^7}{9(2s+a)^4}\cdot\left(44t^5-28t^4-49t^3+10t^2+19t+4\right)\ \left(t=\frac{s}{a}\right)$$

$$=-\frac{16sa^7}{9(2s+a)^4}\cdot(t-1)^2\left(44t^3+60t^2+27t+4\right)<0 \Rightarrow \text{LHS of } (\blacksquare\blacksquare)>0$$

$$\Rightarrow (\blacksquare\blacksquare) \Rightarrow (\blacksquare) \text{ is true } \therefore p_a \geq h_a+\frac{2}{3}\cdot\frac{(b-c)^2}{a} \Rightarrow \frac{p_a}{h_a} \geq 1+\frac{2}{3}\cdot\frac{(b-c)^2}{ah_a}$$

$$=1+\frac{2}{3}\cdot\frac{(b-c)^2}{2F} \Rightarrow \frac{p_a}{h_a} \geq 1+\frac{(b-c)^2}{3F} \text{ and analogs} \Rightarrow$$

$$\sum_{cyc}\frac{p_a}{h_a} \geq 3+\frac{1}{3F}\cdot\sum_{cyc}(b-c)^2 \therefore \frac{p_a}{h_a}+\frac{p_b}{h_b}+\frac{p_c}{h_c} \geq 3+\frac{2}{3}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{F}$$

$\forall\ \Delta\,ABC$, with equality iff $\Delta\,ABC$ is equilateral (QED)