

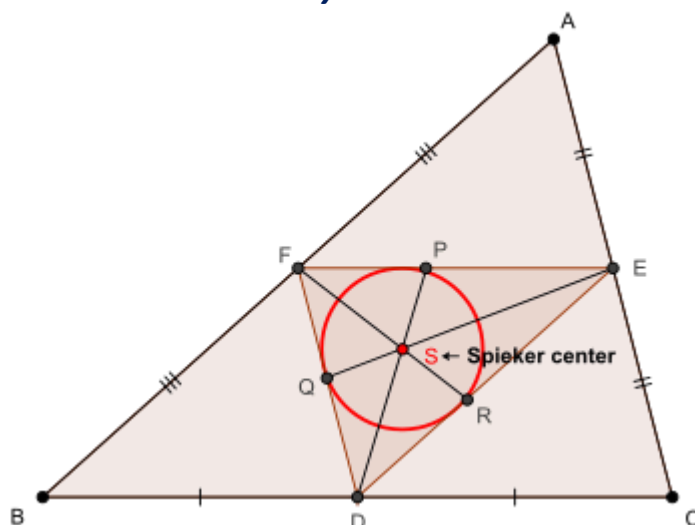
ROMANIAN MATHEMATICAL MAGAZINE

In any ΔABC with p_a, p_b, p_c
 → Spieker cevians, the following relationship holds :

$$p_a + p_b + p_c \leq \frac{14R - r}{3}$$

Proposed by Mohamed Amine Ben Ajiba

Solution by Soumava Chakraborty-Kolkata-India



Let AS produced meet BC at X and $m(\angle BAX) = \alpha$ and $m(\angle CAX) = \beta$ (say)
 and inradius of $\Delta DEF = r'$ (say)

$$\text{Now, } 16[DEF]^2 = 2 \sum \left(\frac{a^2}{4}\right) \left(\frac{b^2}{4}\right) - \sum \frac{a^4}{16} = \frac{1}{16} \left(2 \sum a^2 b^2 - \sum a^4\right) = \frac{16r^2 s^2}{16}$$

$$\Rightarrow [DEF] = \frac{rs}{4} \Rightarrow r' \left(\frac{\frac{a}{2} + \frac{b}{2} + \frac{c}{2}}{2}\right) = \frac{rs}{4} \Rightarrow r' = \frac{r}{2} \rightarrow (1)$$

$$\begin{aligned} \because \text{Spieker center is incenter of } \Delta DEF, \therefore m(\angle AFS) &= B + \frac{C}{2} = \frac{2B + C}{2} = \frac{B + \pi - A}{2} \\ &= \frac{\pi}{2} - \frac{A - B}{2} \text{ and } m(\angle AES) = C + \frac{B}{2} = \frac{\pi}{2} - \frac{A - C}{2} \rightarrow (2) \end{aligned}$$

Via (1), (2) and using cosine law on ΔAFS and ΔAES , we arrive at :

$$\begin{aligned} AS^2 &= \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} \\ &= \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} - \left(\frac{2r}{2 \sin \frac{B}{2}}\right) \left(\frac{b}{2}\right) \sin \frac{A - C}{2} \\ \Rightarrow 2AS^2 &\stackrel{(i)}{=} \frac{r^2}{4 \sin^2 \frac{C}{2}} + \frac{c^2}{4} - \left(\frac{2r}{2 \sin \frac{C}{2}}\right) \left(\frac{c}{2}\right) \sin \frac{A - B}{2} + \frac{r^2}{4 \sin^2 \frac{B}{2}} + \frac{b^2}{4} \end{aligned}$$

$$\begin{aligned}
 & - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 \text{Now, } & \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} + \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & = \frac{r}{2} \left(4R\cos\frac{C}{2} \sin\frac{A-B}{2} + 4R\cos\frac{B}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left(2\sin\frac{A+B}{2} \sin\frac{A-B}{2} + 2\sin\frac{A+C}{2} \sin\frac{A-C}{2} \right) \\
 & = Rr \left(1 - 2\sin^2\frac{B}{2} + 1 - 2\sin^2\frac{C}{2} - 2 \left(1 - 2\sin^2\frac{A}{2} \right) \right) \\
 & = 2Rr \left(\frac{2a(s-b)(s-c) - b(s-c)(s-a) - c(s-a)(s-b)}{abc} \right) \\
 & = \frac{Rr}{8Rs} (2a^3 + (b+c)a^2 - 2a(b^2+c^2) - (b+c)(b-c)^2) \\
 & = \frac{4(b+c)bc\sin^2\frac{A}{2} - 2a \cdot 2bcc\cos A}{8s} = \frac{bc \left((2s-a)\sin^2\frac{A}{2} - a \left(1 - 2\sin^2\frac{A}{2} \right) \right)}{2s} \\
 & = \frac{bc \left((2s+a)\sin^2\frac{A}{2} - a \right)}{2s} = \frac{(2s+a)(s-b)(s-c)}{2s} - 2Rr \\
 & \Rightarrow - \left(\frac{2r}{2\sin\frac{C}{2}} \right) \left(\frac{c}{2} \right) \sin\frac{A-B}{2} - \left(\frac{2r}{2\sin\frac{B}{2}} \right) \left(\frac{b}{2} \right) \sin\frac{A-C}{2} \\
 & \stackrel{(*)}{=} \frac{-(2s+a)(s-b)(s-c)}{2s} + 2Rr \\
 \text{Again, } & \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} = \frac{r^2}{4} \left(\frac{ca}{(s-c)(s-a)} + \frac{ab}{(s-a)(s-b)} \right) \\
 & = \frac{r^2}{4r^2s} (ca(s-b) + ab(s-c)) = \frac{ab+ca}{4} - 2Rr \stackrel{(**)}{=} \frac{r^2}{4\sin^2\frac{B}{2}} + \frac{r^2}{4\sin^2\frac{C}{2}} \\
 & \text{(i), (*), (**)} \Rightarrow 2AS^2 = \frac{b^2+c^2+ab+ca}{4} - \frac{(2s+a)(s-b)(s-c)}{2s} \\
 & = \frac{(a+b+c)(b^2+c^2+ab+ca) - (2a+b+c)(c+a-b)(a+b-c)}{4s} \\
 & = \frac{b^3+c^3-abc+a(2b^2+2c^2-a^2)}{4s} \Rightarrow 2AS^2 \stackrel{(ii)}{=} \frac{b^3+c^3-abc+a(4m_a^2)}{4s} \\
 \text{Via sine law on } \triangle AFS, & \frac{r}{2\sin\frac{C}{2}\sin\alpha} = \frac{AS}{\cos\frac{A-B}{2}} = \frac{4s}{(a+b)\sin\frac{C}{2}} \\
 \Rightarrow c\sin\alpha & \stackrel{(***)}{=} \frac{r(a+b)}{2AS} \text{ and via sine law on } \triangle AES, b\sin\beta \stackrel{(***)}{=} \frac{r(a+c)}{2AS} \\
 \text{Now, } [BAX] + [BAX] & = [ABC] \Rightarrow \frac{1}{2}p_a c\sin\alpha + \frac{1}{2}p_a b\sin\beta = rs \\
 \text{via (***) and (***)} & \Rightarrow \frac{p_a(a+b+a+c)}{4AS} = s \Rightarrow p_a = \frac{4s}{2s+a} AS
 \end{aligned}$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\Rightarrow p_a^2 \stackrel{\text{via (ii)}}{=} \frac{16s^2}{(2s+a)^2} \cdot \frac{b^3 + c^3 - abc + a(4m_a^2)}{8s}$$

$$\therefore p_a^2 \stackrel{(\blacksquare)}{=} \frac{2s}{(2s+a)^2} \left(b^3 + c^3 - abc + a(4m_a^2) \right)$$

Now, $b^3 + c^3 - abc + a(4m_a^2) = b^3 + c^3 + a^3 - abc + a(2b^2 + 2c^2 - a^2) - a^3$

$$= \sum_{\text{cyc}} a^3 - abc + 2a \cdot 2bc \cos A = 2s(s^2 - 6Rr - 3r^2) - 4Rrs + 16Rrs \cos A$$

$$\Rightarrow b^3 + c^3 - abc + a(4m_a^2) \stackrel{(\blacksquare\blacksquare)}{=} 2s(s^2 - 8Rr - 3r^2 + 8Rr \cos A)$$

We have: $\prod_{\text{cyc}} (2s+a) = 8s^3 + 4s^2 \sum_{\text{cyc}} a + 2s \sum_{\text{cyc}} ab + 4Rrs$

$$= 8s^3 + 4s^2 \cdot 2s + 2s(s^2 + 4Rr + r^2) + 4Rrs$$

$$\Rightarrow \prod_{\text{cyc}} (2s+a) \stackrel{(\blacksquare\blacksquare\blacksquare)}{=} 2s(9s^2 + 6Rr + r^2) \text{ and}$$

$$\sum_{\text{cyc}} (2s+b)(2s+c) = \sum_{\text{cyc}} (4s^2 + 2s(2s-a) + bc)$$

$$= 24s^2 - 2s(2s) + s^2 + 4Rr + r^2 \Rightarrow \sum_{\text{cyc}} (2s+b)(2s+c) \stackrel{(\blacksquare\blacksquare\blacksquare\blacksquare)}{=} 21s^2 + 4Rr + r^2$$

$$\stackrel{(\blacksquare), (\blacksquare\blacksquare)}{=} p_a = \frac{2s}{2s+a} \cdot \sqrt{s^2 - 8Rr - 3r^2 + 8Rr \cos A} \stackrel{\text{via } (\blacksquare\blacksquare\blacksquare)}{=} \frac{2s}{2s(9s^2 + 6Rr + r^2)} \cdot \sqrt{s^2 - 8Rr - 3r^2 + 8Rr \cos A} \cdot (2s+b)(2s+c) \text{ and analogs}$$

$$\Rightarrow p_a + p_b + p_c$$

$$= \frac{1}{9s^2 + 6Rr + r^2} \cdot \sum_{\text{cyc}} \left(\sqrt{(s^2 - 8Rr - 3r^2 + 8Rr \cos A)(2s+b)(2s+c)} \cdot \sqrt{(2s+b)(2s+c)} \right)$$

$$\stackrel{\text{CBS}}{\leq} \frac{1}{9s^2 + 6Rr + r^2} \cdot \sqrt{\sum_{\text{cyc}} (s^2 - 8Rr - 3r^2 + 8Rr \cos A)(2s+b)(2s+c)} \cdot \sqrt{\sum_{\text{cyc}} (2s+b)(2s+c)}$$

$$\stackrel{\text{via } (\blacksquare\blacksquare\blacksquare\blacksquare)}{=} \frac{1}{9s^2 + 6Rr + r^2} \cdot \sqrt{\frac{(s^2 - 8Rr - 3r^2)(21s^2 + 4Rr + r^2) + 8Rr \sum_{\text{cyc}} ((8s^2 - 2sa + bc) \cos A)}{}} \cdot \sqrt{21s^2 + 4Rr + r^2}$$

$$= \frac{1}{9s^2 + 6Rr + r^2} \cdot \sqrt{+8Rr \left(8s^2 \cdot \frac{R+r}{R} - 2s \cdot \frac{2rs}{R} + \sum_{\text{cyc}} \left(bc \cdot \frac{b^2 + c^2 - a^2}{2bc} \right) \right)} \cdot \sqrt{21s^2 + 4Rr + r^2}$$

$$\left(\because \sum_{\text{cyc}} a \cos A = \frac{2rs}{R} \right)$$

$$= \frac{1}{9s^2 + 6Rr + r^2} \cdot \sqrt{+8r(8(R+r)s^2 - 4rs^2 + R(s^2 - 4Rr - r^2))} \cdot \sqrt{21s^2 + 4Rr + r^2}$$

$$\Rightarrow (p_a + p_b + p_c)^2 \leq$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\frac{(21s^2 + 4Rr + r^2)(21s^4 - (92Rr + 30r^2)s^2 - r^2(64R^2 + 28Rr + 3r^2))}{(9s^2 + 6Rr + r^2)^2} \stackrel{?}{\leq} \frac{(14R - r)^2}{9}$$

$$\Leftrightarrow \boxed{\begin{aligned} &3969s^6 - (15876R^2 + 14364Rr + 5562)s^4 \\ &-rs^2(21168R^3 + 15912R^2r + 6804Rr^2 + 855r^3) \\ &-r^2(7056R^4 + 3648R^3r + 1480R^2r^2 + 344Rr^3 + 28r^4) \stackrel{?}{\geq} 0 \end{aligned}} \quad (*)$$

Now, Rouché $\Rightarrow s^2 - (m - n) \geq 0$ and $s^2 - (m + n) \leq 0$, where

$$m = 2R^2 + 10Rr - r^2 \text{ and } n = 2(R - 2r) \cdot \sqrt{R^2 - 2Rr}$$

$$\therefore (s^2 - (m + n))(s^2 - (m - n)) \leq 0$$

$$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3 \stackrel{(\heartsuit)}{\leq} 0$$

$$\therefore 3969s^2(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3) \leq 0 \Rightarrow \text{in order}$$

to prove $(*)$, it suffices to prove : LHS of $(*) \leq$

$$3969s^2(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3)$$

$$\Leftrightarrow \boxed{\begin{aligned} &(16254R - 3375r)s^4 - s^2(68796R^3 + 51606R^2r + 13608Rr^2 + 1206r^2) \\ &-r(1764R^4 + 912R^3r + 370R^2r^2 + 86Rr^3 + 7r^4) \stackrel{?}{\leq} 0 \end{aligned}} \quad \text{and} \quad (**)$$

$$\therefore (16254R - 3375r)(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3) \stackrel{\text{via } (\heartsuit)}{\leq} 0$$

\therefore in order to prove $(**)$, it suffices to prove : LHS of $(**)$ \leq

$$(16254R - 3375r)(s^4 - (4R^2 + 20Rr - 2r^2)s^2 + r(4R + r)^3)$$

$$\Leftrightarrow \boxed{\begin{aligned} &(1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3)s^2 \\ &+r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4) \stackrel{(***)}{\geq} 0 \end{aligned}}$$

Case 1 $1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3 \geq 0$ and then : LHS of $(***)$
 $\geq r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4) > 0$

$\Rightarrow (***)$ is true

Case 2 $1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3 < 0$ and then : LHS of $(***)$

$$= - \left(-(1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3) \right) s^2$$

$$+r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4)$$

$$\stackrel{\text{Gerretsen}}{\geq} - \left(-(1890R^3 - 129987R^2r + 56808Rr^2 - 2772r^3) \right) (4R^2 + 4Rr + 3r^2)$$

$$+r(521010R^4 + 282552R^3r + 16709R^2r^2 - 12080Rr^3 - 1684r^4) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 3780t^5 + 1287t^4 - 8439t^3 - 85538t^2 + 65924t - 3704 \stackrel{?}{\geq} 0 \quad \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(3780t^4 + 11871t^3 + 3713t^2 + 17782t(t - 2) + 2500) \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$\therefore t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (***)$ is true \therefore combining both cases, $(***) \Rightarrow (**)$ \Rightarrow $(*)$

$$\text{is true } \forall \Delta ABC \Rightarrow (p_a + p_b + p_c)^2 \leq \frac{(14R - r)^2}{9}$$

$$\therefore p_a + p_b + p_c \leq \frac{14R - r}{3} \quad \forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$