ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{\cot A\cot B\cot C}{\sin A\sin B\sin C}\le\frac{8}{27}$$

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$$\boldsymbol{Note}: \boldsymbol{In}\ \Delta\ \boldsymbol{ABC}\ , A+B+C=\pi, \sin 2A+\sin 2B+\sin 2C$$
$$=2\sin(A+B)\cos(A-B)+\sin 2C$$
$$=2\sin C\ [\cos(A-B)-\cos(A+B)]=4\sin A\sin B\sin C.$$

$\boldsymbol{Case}\ 1\ \boldsymbol{Let\ the\ triangle\ acute, then}\ 4\sin A\sin B\sin C.\overset{AM-GM}{\ge}3\sqrt[3]{\sin 2A.\sin 2B.\sin 2C}\ \boldsymbol{or}$

$$\frac{4}{3}\ge\sqrt[3]{\frac{\sin 2A.\sin 2B.\sin 2C}{\sin^3 A.\sin^3 B\sin^3 C}}=2\sqrt[3]{\left(\frac{\cot A\cot B\cot C}{\sin A\sin B\sin C}\right)}\ \boldsymbol{or}\ \frac{2}{3}\ge\sqrt[3]{\frac{\cot A\cot B\cot C}{\sin A\sin B\sin C}}\ \boldsymbol{or}$$
$$\frac{8}{27}\ge\frac{\cot A\cot B\cot C}{\sin A\sin B\sin C}$$

$\boldsymbol{Case 2\ for\ non\ acute\ triangle}\ \prod\cot A<0\ \mathit{and}$

$$\prod\sin A>0\ \mathit{so\ the\ given\ expression}<0<\frac{8}{27}, \boldsymbol{equality\ for}\ A=B=C=\frac{\pi}{3}$$