ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)\le\left(\frac{1+\sqrt{3}}{2}\right)^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)\le\left(\frac{1+\sqrt{3}}{2}\right)^3$$

$$\Leftrightarrow\sum_{cyc}\ln(\sin A+\cos A)\overset{(*)}{\le}3\ln\left(\frac{1+\sqrt{3}}{2}\right)$$

Let $f(x)=\ln(\sin x+\cos x)\ \forall x\in(0,\pi)$ and then : $f''(x)=\frac{-2}{(\sin x+\cos x)^2}<0$

$$\Rightarrow f(x)\text{ is concave}\ \therefore\sum_{cyc}\ln(\sin A+\cos A)\overset{\text{Jensen}}{\le}3\ln\left(\sin\frac{\pi}{3}+\cos\frac{\pi}{3}\right)$$

$$=3\ln\left(\frac{1+\sqrt{3}}{2}\right)\Rightarrow(*)\text{ is true}\ \therefore(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)$$

$$\le\left(\frac{1+\sqrt{3}}{2}\right)^3\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

Solution 2 by Tapas Das-India

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)\overset{AM-GM}{\le}\left(\frac{\sum\sin A+\sum\cos A}{3}\right)^3=$$

$$=\left(\frac{\frac{s}{R}+\frac{r}{R}+1}{3}\right)^3\overset{MITRINOVIC}{\le}\left(\frac{\frac{3\sqrt{3}R}{2R}+\frac{r}{R}+1}{3}\right)^3\overset{EULER}{\le}\left(\frac{\frac{3\sqrt{3}}{2}+\frac{1}{2}+1}{3}\right)^3=\left(\frac{\sqrt{3}+1}{2}\right)^3$$

Equality holds for $a=b=c$.