ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}\geq\frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnem

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\text{LHS}=\sum_{\text{cyc}}\frac{a^2}{a.\sqrt{a^2+3bc}}\overset{\text{Bergstrom}}{\geq}\frac{4s^2}{\sum_{\text{cyc}}\sqrt{a}.\sqrt{a^3+3abc}}\overset{\text{CBS}}{\geq}$$

$$\frac{4s^2}{\sqrt{2s}.\sqrt{2s(s^2-6Rr-3r^2)+36Rrs}}\overset{?}{\geq}\frac{3}{2}\Leftrightarrow 7s^2\underset{(*)}{\overset{?}{\geq}}108Rr-27r^2$$

$$\text{Now, }7s^2\overset{\text{Gerretsen}}{\geq}112Rr-35r^2=108Rr-27r^2+4r(R-2r)\overset{\text{Euler}}{\geq}108Rr-27r^2$$

$$\Rightarrow(*)\text{ is true }\therefore\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}\geq\frac{3}{2}$$

$\forall\,\Delta$ ABC,'' ='' iff Δ ABC is equilateral (QED)

Solution 2 by Tapas Das-India

$$\text{In any }\Delta ABC\text{ prove that }\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}\geq\frac{3}{2}\;(\text{Nguyen Hung Cuong})$$

$$\text{solution: }\frac{a}{\sqrt{a^2+3bc}}+\frac{b}{\sqrt{b^2+3ca}}+\frac{c}{\sqrt{c^2+3ab}}$$

$$=\sum\frac{a^{\frac{3}{2}}}{\sqrt{a^3+3abc}}\geq\frac{(a+b+c)^{\frac{3}{2}}}{\sqrt{a^3+b^3+c^3+9abc}}\;(\text{Radon})$$

$$=\frac{(2s)^{\frac{3}{2}}}{\sqrt{(2s)(s^2+12Rr-3r^2)}}$$

$$=\frac{2s}{\sqrt{s^2+12Rr-3r^2}}\text{, now we need to show}$$

$$\frac{2s}{\sqrt{s^2+12Rr-3r^2}}\geq\frac{3}{2}\text{ or}$$

$$7s^2\geq 108Rr-27r^2\text{ or }R\overset{\text{Gerretsen}}{\geq}2r\;(\text{Euler})$$