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In any Δ ABC, the following relationship holds :

$$\frac{l_a+l_b+l_c}{R}\le 2\left(\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\right)$$

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Solution by Soumava Chakraborty-Kolkata-India

$$2\left(\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\right)=2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\sum_{cyc}\frac{1}{\cos\frac{A}{2}}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{2s}{4R}.\frac{9}{\sum_{cyc}\cos\frac{A}{2}}\overset{\text{Jensen}}{\ge}\frac{2s}{4R}.\frac{9}{3.\frac{\sqrt{3}}{2}}$$

$$\left(\because f(x)=\cos\frac{x}{2}\ \forall\, x\in(0,\pi)\Rightarrow f''(x)=\frac{-\cos\frac{x}{2}}{4}<0\Rightarrow f(x)\text{ is concave}\right)=\frac{\sqrt{3}s}{R}$$

$$\ge\frac{l_a+l_b+l_c}{R}\left(\because l_a\le\sqrt{s(s-a)}\text{ and analogs}\Rightarrow\sum_{cyc}l_a\overset{\text{CBS}}{\le}\sqrt{3}.\sqrt{\sum_{cyc}s(s-a)}=\sqrt{3}s\right)$$

$$\therefore\frac{l_a+l_b+l_c}{R}\le 2\left(\cos\frac{A}{2}\cos\frac{B}{2}+\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{C}{2}\cos\frac{A}{2}\right)$$

$\forall\,\Delta$ ABC, " = " iff Δ ABC is equilateral (QED)