ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\geq 4$$

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$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\geq 4$$

$$\Leftrightarrow (3a+b)(2b+a)(2c+b)+(3b+c)(2c+b)(2a+c)$$
$$+(3c+a)(2a+c)(2b+a)\geq 4(2a+c)(2b+a)(2c+b)$$

$$\overset{\text{expanding and re-arranging}}{\Leftrightarrow} 2\sum_{\text{cyc}}a^3+6abc\geq\sum_{\text{cyc}}a^2b+3\sum_{\text{cyc}}ab^2$$

$$\Leftrightarrow 2\sum_{\text{cyc}}(y+z)^3+6(x+y)(y+z)(z+x)$$

$$\geq\sum_{\text{cyc}}\left((y+z)^2(z+x)\right)+3\sum_{\text{cyc}}\left((y+z)(z+x)^2\right)$$

$$(a=y+z, b=z+x, c=x+y)\quad\overset{\text{expanding and re-arranging}}{\Leftrightarrow}$$

$$\sum_{\text{cyc}}x^2y+3\sum_{\text{cyc}}xy^2\geq 12xyz\to\text{true}\because\sum_{\text{cyc}}x^2y\overset{A-G}{\geq}3xyz\text{ and }\sum_{\text{cyc}}xy^2\overset{A-G}{\geq}3xyz$$

$$\therefore\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\geq 4\ \forall\ \Delta\text{ ABC}, ''='' \text{ iff }\Delta\text{ ABC is equilateral (QED)}$$