ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\cos^2\left(\frac{A-B}{2}\right)+\cos^2\left(\frac{B-C}{2}\right)+\cos^2\left(\frac{C-A}{2}\right)\ge 24\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

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Lemma 1: In ΔABC the following relationship holds:

$$\cos^2\left(\frac{A-B}{2}\right)+\cos^2\left(\frac{B-C}{2}\right)+\cos^2\left(\frac{C-A}{2}\right)=\frac{s^2+r^2+2Rr}{4R^2}+1$$

Proof:

$$\cos^2\left(\frac{A-B}{2}\right)+\cos^2\left(\frac{B-C}{2}\right)+\cos^2\left(\frac{C-A}{2}\right)=\sum_{cyc}\cos^2\left(\frac{A-B}{2}\right)=$$

$$=\sum_{cyc}\frac{1+\cos(A-B)}{2}=\frac{3}{2}+\frac{1}{2}\sum_{cyc}\cos A\cos B+\frac{1}{2}\sum_{cyc}\sin A\sin B=$$

$$=\frac{3}{2}+\frac{s^2+r^2-4R^2}{8R^2}+\frac{1}{2}\sum_{cyc}\frac{ab}{4R^2}=$$

$$=\frac{12R^2+s^2+r^2-4R^2+s^2+r^2+4Rr}{8R^2}=\frac{s^2+r^2+2Rr}{4R^2}+1$$

Lemma 2: In ΔABC the following relationship holds:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{4R}$$

Proof:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\prod_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}}=$$

$$=\frac{(s-a)(s-b)(s-c)}{abc}=\frac{s(s-a)(s-b)(s-c)}{sabc}=\frac{F^2}{s\cdot 4RF}=$$

$$=\frac{F}{4Rs}=\frac{rs}{4Rs}=\frac{r}{4R}$$

Using Lemma 1 and Lemma 2 we must prove that:

$$\frac{s^2+r^2+2Rr}{4R^2}+1\ge 24\cdot\frac{r}{4R}$$

$$s^2 + r^2 + 2Rr + 4R^2 \geq 24Rr$$

$s^2 \geq 22Rr - r^2 - 4R^2$ **(to prove)**

$$s^2 \overset{GERRETSEN}{\geq} 16Rr - 5r^2 \geq 22Rr - r^2 - 4R^2 \Leftrightarrow$$

$$\Leftrightarrow 4R^2 - 6Rr - 4r^2 \geq 0 \Leftrightarrow 2R^2 - 3Rr - 2r^2 \geq 0$$

$$2R^2 - 4Rr + Rr - 2r^2 \geq 0$$

$$2R(R - 2r) + r(R - 2r) \geq 0$$

$$(R - 2r)(2R + r) \geq 0$$

$$R - 2r \geq 0$$
$$R \geq 2r \ (Euler)$$

Equality holds for $a = b = c$.