In ΔABC the following relationship holds:

$$\left(m_a^2+m_b^2\right)\cos C+\left(m_b^2+m_c^2\right)\cos A+\left(m_c^2+m_a^2\right)\cos B\le\frac{27R^2}{4}$$

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$$m_a^2+m_b^2=\frac{a^2+b^2+4c^2}{4}=\frac{(a^2+b^2+c^2)+3c^2}{4}$$

$$WLOG\ a\ge b\ge c\ ,\cos A\le\cos B\le\cos C$$

$$\left(m_a^2+m_b^2\right)\cos C+\left(m_b^2+m_c^2\right)\cos A+\left(m_c^2+m_a^2\right)\cos B=$$

$$=\sum\frac{(a^2+b^2+c^2)+3c^2}{4}\cos C=$$

$$=\frac{1}{4}\left(\sum a^2\right)\left(\sum\cos C\right)+\frac{3}{4}\left(\sum c^2\cos C\right)\overset{CEBYSHEV}{\le}$$

$$\le\frac{1}{4}\left(\sum a^2\right)\left(\sum\cos C\right)+\frac{3}{4}\frac{1}{3}\left(\sum a^2\right)\left(\sum\cos C\right)=$$

$$=\frac{1}{2}\left(\sum a^2\right)\left(\sum\cos C\right)\overset{LEIBNIZ}{\le}\frac{1}{2}.9R^2\left(1+\frac{r}{R}\right)\le$$

$$\le\frac{9R^2}{2}\left(1+\frac{R}{2R}\right)=\frac{27R^2}{4}(Euler)$$

Equality holds for: $a=b=c$.