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In ΔABC the following relationship holds:

$$\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\le\frac{5}{8}+\frac{r}{4R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum ab=s^2+r^2+4Rr\overset{Gerretsen}{\le}4(R+r)^2\quad(1)$$

$$\sum\frac{1}{\sin\frac{A}{2}}=\sum\sqrt{\frac{bc}{(s-b)(s-c)}}\overset{CBS}{\le}\sqrt{ab+bc+ca}.\sqrt{\sum\frac{1}{(s-b)(s-c)}}\overset{(1)}{\le}2(R+r).\frac{1}{r}$$

$$\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}=\prod\sin\frac{A}{2}.\sum\frac{1}{\sin\frac{A}{2}}\le$$
$$\le\frac{r}{4R}.\frac{2(R+r)}{r}=\frac{1}{2}+\frac{r}{2R}=\frac{5}{8}-\frac{1}{8}+\frac{r}{2R}=$$

$$=\frac{5}{8}-\frac{1}{4}.\frac{1}{2}+\frac{r}{2R}\le\frac{5}{8}+\frac{r}{2R}-\frac{r}{4R}(Euler)=\frac{5}{8}+\frac{r}{4R}$$