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In any Δ ABC, the following relationship holds :

$$1+\frac{r}{R}\le \sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\sqrt{2+\frac{r}{2R}}$$

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$$\sum_{cyc}\operatorname{cosec}\frac{A}{2}=\sqrt{\frac{bc(s-a)}{(s-b)(s-c)(s-a)}}=\frac{1}{r.\sqrt{s}}.\sum_{cyc}\sqrt{bc(s-a)}\overset{CBS}{\le}$$

$$\frac{1}{r.\sqrt{s}}.\sqrt{s^2+4Rr+r^2}.\sqrt{\sum_{cyc}(s-a)}\overset{Gerretsen}{\le}\frac{\sqrt{s}}{r.\sqrt{s}}.\sqrt{4R^2+8Rr+4r^2}=\frac{\sqrt{4(R+r)^2}}{r}$$

$$\therefore\sum_{cyc}\operatorname{cosec}\frac{A}{2}\le\frac{2R+2r}{r}\to(1)$$

$$\text{Now,}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2=\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$=\frac{2R-r}{2R}+\frac{2r}{4R}.\sum_{cyc}\operatorname{cosec}\frac{A}{2}\overset{via\ (1)}{\le}\frac{2R-r}{2R}+\frac{2r}{4R}.\frac{2R+2r}{r}=\frac{4R+r}{2R}=2+\frac{r}{2R}$$

$$\Rightarrow\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\sqrt{2+\frac{r}{2R}}$$

$$\text{Again,}\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}=\sum_{cyc}\frac{2\cos\frac{B+C}{2}.\cos\frac{B-C}{2}}{2\cos\frac{B-C}{2}}\overset{\because 0<\cos\frac{B-C}{2}\le 1\text{ and analogs}}{\ge}$$

$$\sum_{cyc}\frac{2\cos\frac{B+C}{2}.\cos\frac{B-C}{2}}{2}=\sum_{cyc}\frac{\cos B+\cos C}{2}=\sum_{cyc}\cos A=1+\frac{r}{R}\therefore\sum_{cyc}\sin\frac{A}{2}\ge 1+\frac{r}{R}$$

$$\text{and so,}1+\frac{r}{R}\le\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\sqrt{2+\frac{r}{2R}}$$

$\forall\,\Delta$ ABC, "$=$" iff Δ ABC is equilateral (QED)