ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC non-right angled holds:

$$,\frac{\tan A}{\tan B}+\frac{\tan B}{\tan C}+\frac{\tan C}{\tan A}\ge\frac{\sin 2A}{\sin 2B}+\frac{\sin 2B}{\sin 2C}+\frac{\sin 2C}{\sin 2A}$$

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$$Lemma\ a,b,c>0\ then\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}$$

$Proof$:

$$WLOG\ c=\min(a,b,c)\,,now:$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\ge\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}.-3=\frac{1}{ab}(a-b)^2+\frac{1}{ac}(a-c)(b-c)\ge$$

$$\ge\left(\frac{1}{ab}-\frac{1}{(a+c)(b+c)}\right)(a-b)^2+\left(\frac{1}{ac}-\frac{1}{(a+c)(a+b)}\right)\times(a-c)(b-c)\ge 0\,,$$

$$since\ c=\min(a,b,c)$$

$$LHS\overset{lemma}{\ge}\sum\frac{\tan A+\tan B}{tanB+tanC}=\sum\frac{\sin(B+A)}{\sin(B+C)}.\frac{cosC}{cosA}=$$

$$=\sum\frac{2sinC\ cosC}{2\sin A\ cosA}=\sum\frac{sin2C}{sin2A}\ \ (Note: A+B+C=\pi)$$

Equality holds for $A=B=C$.