ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{a+b}{h_c}+\frac{b+c}{h_a}+\frac{c+a}{h_b}\geq 4\sqrt{3}$$

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Solution by Daniel Sitaru-Romania

$$\frac{a+b}{h_c}+\frac{b+c}{h_a}+\frac{c+a}{h_b}=\sum_{cyc}\frac{a+b}{h_c}=\sum_{cyc}\frac{a+b}{\frac{2F}{c}}=$$

$$=\frac{1}{2F}\sum_{cyc}(ac+bc)=\frac{1}{F}\sum_{cyc}ab=\frac{1}{F}\cdot(s^2+r^2+4Rr)\overset{GERRETSEN}{\geq}$$

$$\geq\frac{1}{rs}\cdot(16Rr-5r^2+r^2+4Rr)=\frac{1}{s}\cdot(20R-4r)\overset{EULER}{\geq}$$

$$\geq\frac{1}{s}\cdot\left(20R-4\cdot\frac{R}{2}\right)=\frac{18R}{s}\overset{MITRINOVIC}{\geq}\frac{18R}{\frac{3\sqrt{3}}{2}\cdot R}=4\sqrt{3}$$

Equality holds for $a=b=c$.