ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\frac{\cos^{2024} A}{\cos^{2022} B+\cos^{2022} C}+\frac{\cos^{2024} B}{\cos^{2022} C+\cos^{2022} A}+\frac{\cos^{2024} C}{\cos^{2022} A+\cos^{2022} B}\ge\frac{3}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\cos^{2024} A}{\cos^{2022} B+\cos^{2022} C}+\frac{\cos^{2024} B}{\cos^{2022} C+\cos^{2022} A}+\frac{\cos^{2024} C}{\cos^{2022} A+\cos^{2022} B}$$
$$=\sum_{cyc}\frac{\cos^2 A\left(\sum_{cyc}\cos^{2022} A-\left(\cos^{2022} B+\cos^{2022} C\right)\right)}{\cos^{2022} B+\cos^{2022} C}$$
$$=\left(\sum_{cyc}\cos^{2022} A\right)\left(\sum_{cyc}\frac{\cos^2 A}{\cos^{2022} B+\cos^{2022} C}\right)-\sum_{cyc}\cos^2 A\overset{\text{Chebyshev}}{\ge}$$
$$\frac{1}{3}\left(\sum_{cyc}\cos^{2022} A\right)\left(\sum_{cyc}\cos^2 A\right)\left(\sum_{cyc}\frac{1}{\cos^{2022} B+\cos^{2022} C}\right)-\sum_{cyc}\cos^2 A$$
$$\left(\begin{array}{c}\because \text{WLOG assuming } a\ge b\ge c\Rightarrow \cos^2 A\le\cos^2 B\le\cos^2 C \text{ and}\\ \frac{1}{\cos^{2022} B+\cos^{2022} C}\le\frac{1}{\cos^{2022} C+\cos^{2022} A}\le\frac{1}{\cos^{2022} A+\cos^{2022} B}\end{array}\right)$$
$$\overset{\text{Bergstrom}}{\ge}\frac{1}{3}\left(\sum_{cyc}\cos^{2022} A\right)\left(\sum_{cyc}\cos^2 A\right)\left(\frac{9}{2\sum_{cyc}\cos^{2022} A}\right)-\sum_{cyc}\cos^2 A$$
$$\left(\begin{array}{c}\text{we note that : }\cos^{2022} B+\cos^{2022} C\ge\frac{1}{2^{2021}}(\cos B+\cos C)^{2022}\\ =\frac{1}{2^{2021}}\left(2\sin\frac{A}{2}\cos\frac{B-C}{2}\right)^{2022}\overset{0<\cos\frac{B-C}{2}\le 1}{>}0\end{array}\right)$$
$$=\frac{1}{2}\sum_{cyc}\cos^2 A=\frac{1}{2}\left(3-\frac{\sum_{cyc}a^2}{4R^2}\right)\overset{\text{Leibnitz}}{\ge}\frac{1}{2}\left(3-\frac{9R^2}{4R^2}\right)$$
$$\Rightarrow\frac{\cos^{2024} A}{\cos^{2022} B+\cos^{2022} C}+\frac{\cos^{2024} B}{\cos^{2022} C+\cos^{2022} A}+\frac{\cos^{2024} C}{\cos^{2022} A+\cos^{2022} B}\ge\frac{3}{8}$$

$\forall\,\Delta$ ABC," = " iff Δ ABC is equilateral (QED)